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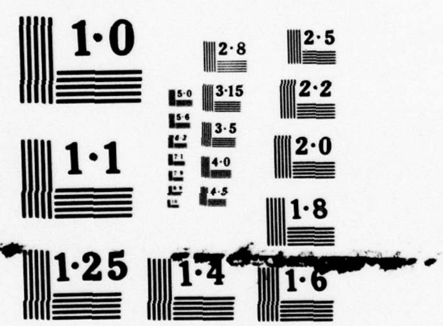

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This report presents the results of an application of Disturbance Accommodating Controller design techniques to three example control problems involving unmeasurable external disturbances of the waveform-mode type.

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1. INTRODUCTION

This report describes the results of a study of Disturbance Accommodating Controller (DAC) design. The DAC method of design, using a combination of waveform-mode disturbance modeling and state-variable control techniques, was developed by Dr. C. D. Johnson of the University of Alabama in Huntsville[1,2,3,4]. As a tool for design of controllers, the DAC approach permits three primary modes of disturbance accommodation: (1) cancellation (absorption) of disturbance effects, (2) minimization of disturbance effects, or (3) constructive utilization of the disturbances as an aid in accomplishing the primary control task. These disturbance accommodations are realized in addition to the usual control efforts required to satisfy system performance requirements without disturbance.

A large number of papers have been published which describe the general theory and procedures involved in the use of DAC technique. However, only a few practical numerical examples have been worked out so far. The purpose of the present report is to carry out detailed numerical designs of three DAC example problems. The results presented in this report were obtained using the disturbance cancellation mode of DAC design on three linear, time-invariant plants. The controllers obtained via this design method are termed "Disturbance Absorber" controllers.

It is not the intent of this report to thoroughly cover all the background theory involved in the development of DAC design procedures. This background theory can best be obtained by reading the original papers; see, for instance, References 1-4.

2. SOME BACKGROUND

This report considers controlled systems (plants) which can be described by the state equations of the form

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} + \underline{F}\,\underline{w}$$

$$\underline{y} = \underline{C}\,\underline{x}$$

where

$\underline{x}$ is the plant state vector,

$\underline{u}$ is the plant control input vector,

$\underline{w}$ is the vector of external disturbance acting on the plant,

$\underline{y}$ is the system output vector, and

$\underline{A}$, $\underline{B}$, $\underline{F}$, $\underline{C}$ are appropriate size, known matrices which are assumed constant.

The class of external disturbances w(t) for which the DAC theory is intended is characterized by the presence of "waveform structure" in w(t), i.e., the functions w(t) can be described by known differential equations which the w(t) satisfy "almost everywhere". For purposes of this study, the disturbances considered will be assumed to be described by the following general set of linear disturbance state equations:

$$\underline{w} = \underline{H}\,\underline{z} + \underline{L}\,\underline{x}$$

$$\dot{\underline{z}} = \underline{D}\,\underline{z} + \underline{M}\,\underline{x} + \underline{\sigma}$$

where

$\underline{z}$ is the disturbance "state" vector,

$\underline{\sigma}$ is a sequence of randomly arriving vector impulses, and

$\underline{D}$, $\underline{H}$, $\underline{L}$, $\underline{M}$ are known, time-invariant matrices.

It should be noted that $\underline{L}$ and $\underline{M}$ allow for "state dependency" of $\underline{w}$.

In most practical applications, neither the complete set of system (plant) state variables nor the various components $\underline{w}_i(t)$ of the disturbance are available for direct on-line measurement. Therefore, the DAC's we seek to design are restricted to operate only on information in the available on-line measurements of the system outputs, set-points and servo-commands and any disturbance components which may actually be available for direct measurement. One of the most important features of the DAC technique is that it can effectively handle disturbances which cannot be directly measured.

Since the idealized DAC control law is a function of the real time system state, $\underline{x}$ and disturbance state, $\underline{z}$, the required on-line data for practical implementation of a DAC must be generated via use of state constructors (observers) operating on real time system outputs $\underline{y}$ and control inputs $\underline{u}$. Since the external disturbances $\underline{w}(t)$ are assumed to have waveform structure and to be modeled by known linear state models, a state constructor can be designed to generate estimates $\underline{\hat{z}}$ of the instantaneous disturbance state $\underline{z}$. In addition, that same state constructor can be designed to produce estimates $\underline{\hat{x}}$ of the instantaneous system state $\underline{x}$.

Thus, the designer can utilize such a composite-type state constructor to implement DAC control laws in the form

$$u = f\,(\hat{x}, \hat{z}, t).$$

Of course, for acceptable performance the real time estimation errors

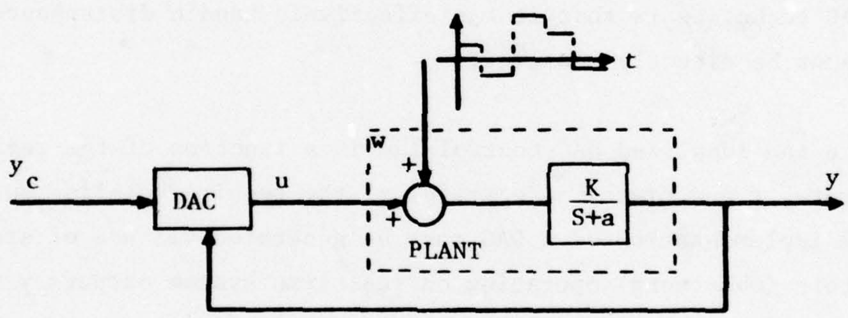
$$\underline{\varepsilon}_x = \underline{x} - \hat{\underline{x}}$$

$$\underline{\varepsilon}_z = \underline{z} - \hat{\underline{z}}$$

must settle to zero rapidly in comparison to system settling times.

3. EXAMPLE DESIGN PROBLEM NO. 1

The first example design problem we will consider is a "servo problem" involving a first-order controlled plant acted upon by a random-like piecewise constant external disturbance, as shown below.



The disturbance $\underline{w}(t)$ is not known a priori and cannot be directly measured. The servo command y_c is also assumed to be piecewise constant and unknown a priori, but is assumed measurable in a real time fashion. The objective of the control design is to create a controller which will accomplish high-fidelity servo-tracking of y_c while, at the same time, consistently counteracting the effects of $\underline{w}$. For this purpose, two separate DAC designs are to be considered: Case (A) - A design utilizing a full-dimensional composite state reconstructor, and Case (B) - A design utilizing a reduced-order composite state reconstructor.

A. CASE A: DAC DESIGN WITH A FULL-DIMENSIONAL COMPOSITE STATE RECONSTRUCTOR.

Piecewise constant disturbances $\underline{w}(t)$ can be accurately modeled by setting (see Reference 1),

$$\rho = p$$

where ρ is the dimension of $\underline{z}$ and p is the dimension of $\underline{w}$.

$$\underline{L} = \underline{D} = \underline{M} = \underline{0}$$

$$\underline{H} = \underline{I}$$

Thus, in this case

$$\underline{w} = \underline{z}$$

$$\dot{\underline{z}} = \underline{\sigma}(t)$$

For the given plant, let $u' = u + w$.
Then,

$$\frac{y}{u'} = \frac{K}{s+a}$$

$$(s+a)y = Ku'$$

$$y = \frac{1}{s}\,[Ku' - ay]$$

Thus, letting

$$x_1 = y$$

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$$\dot{x}_1 = Ku' - ay = Ku' - ax_1$$

therefore

$$\dot{x}_1 = Ku + Kw - ax_1$$

so

$$\left.\begin{aligned} \dot{\underline{x}} &= \underline{A}\,\underline{x} + \underline{B}\,\underline{u} + \underline{F}\,\underline{w} = -\,ax_1 + Ku + Kw \\ \underline{y} &= \underline{C}\,\underline{x} = x_1\,. \end{aligned}\right\} \quad (1)$$

Hence

$$\underline{A} = [-a]$$

$$\underline{B} = \underline{F} = [K]$$

$$\underline{C} = [1]\,.$$

Following the ideas in Reference 1, we agree to split the control $\underline{u}$ into $\underline{u}_c + \underline{u}_R$ where $\underline{u}_c$ is that part of the control which will be designed to counteract the disturbance and $\underline{u}_R$ is that part which will regulate the system.

We must first check for the existence of $\underline{u}_c$ as shown in Reference 1. The control $\underline{u}_c$ will exist if, and only if,

$$\underline{F} \equiv \underline{B}\,\underline{\Gamma} \text{ for some matrix } \underline{\Gamma}.$$

However, in this particular example

$$\underline{B} = \underline{F} = K \text{ is a scalar.}$$

Thus, $\underline{u}_c$ exists ($\Gamma = 1$) and u_c can be designed as (Reference 1)

$$\underline{u}_c = -\underline{\Gamma}\,\underline{w} = -\underline{w} \quad . \tag{2}$$

For the design of the full-dimensional composite state reconstructor, we will use the following form proposed in Reference 1

$$\begin{pmatrix} \dot{\hat{\underline{x}}} \\ \dot{\hat{\underline{z}}} \end{pmatrix} = \left[\begin{array}{c|c} \underline{A} + \underline{F}\,\underline{L} + \underline{K}_1\,\underline{C} & \underline{F}\,\underline{H} \\ \hline M + \underline{K}_2\,\underline{C} & \underline{D} \end{array}\right] \begin{pmatrix} \hat{\underline{x}} \\ \hat{\underline{z}} \end{pmatrix} - \left[\begin{array}{c} \underline{K}_1 \\ \hline \underline{K}_2 \end{array}\right] \underline{y} + \left[\begin{array}{c} \underline{B} \\ \hline 0 \end{array}\right] \underline{u}_R \quad . \tag{3}$$

The gain matrices $\underline{K}_1$ and $\underline{K}_2$ are designed by examining the error dynamics which are known to obey (Reference 2)

$$\dot{\underline{\varepsilon}} = \left[\begin{array}{c|c} \underline{A} + \underline{F}\,\underline{L} + \underline{K}_1\,\underline{C} & \underline{F}\,\underline{H} \\ \hline \underline{M} + \underline{K}_2\,\underline{C} & \underline{D} \end{array}\right] (\underline{\varepsilon}) + \left(\begin{array}{c} 0 \\ \hline \underline{\sigma} \end{array}\right) \quad . \tag{4}$$

For this example, the preceeding expression reduces to

$$\dot{\underline{\varepsilon}} = \left[\begin{array}{c|c} -a + k_1 & K \\ \hline k_2 & 0 \end{array}\right] (\underline{\varepsilon}) + \left(\begin{array}{c} 0 \\ \hline \underline{\sigma} \end{array}\right) \quad . \tag{5}$$

Let

$$\underline{\tilde{A}} = \left[\begin{array}{c|c} -a + k_1 & K \\ \hline k_2 & 0 \end{array}\right] = \text{characteristic matrix of the } \dot{\underline{\varepsilon}} \text{ dynamics.}$$

It is desired that $\varepsilon(t) \rightarrow 0$ rapidly which means the roots of the characteristic polynomial should be "large" negative numbers. The eigenvalues of $\underline{\tilde{A}}$ are found as the roots, λ_i, of the characteristic polynomial

$$\det\left[\underline{\tilde{A}} - \lambda\,\underline{I}\right] = 0$$

or

$$\det \left[\left[\begin{array}{c|c} k_1 - a & K \\ \hline k_2 & 0 \end{array} \right] - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right] = 0$$

or

$$\det \left[\begin{array}{c|c} k_1 - a - \lambda & K \\ \hline k_2 & -\lambda \end{array} \right] = 0$$

or

$$\lambda^2 + (a - k_1)\lambda - k_2 K = 0 \quad . \tag{6}$$

Let the desired roots of this polynomial be λ_1, λ_2. Thus, the desired characteristic equation is

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

or

$$\lambda^2 + (- \lambda_1 - \lambda_2)\lambda + \lambda_1 \lambda_2 = 0 \quad . \tag{7}$$

Comparing this with Equation (6) we see that the unknowns k_1, k_2 must satisfy

$$\lambda_1 \lambda_2 = - k_2 K$$

$$- \lambda_1 - \lambda_2 = a - k_1$$

therefore,

$$\left.\begin{aligned} k_1 &= a + \lambda_1 + \lambda_2 \\ k_2 &= - \frac{\lambda_1 \lambda_2}{K} \; . \end{aligned}\right\} \quad (8)$$

Choosing the roots λ_1, λ_2 properly will insure rapid settling out of the error transients.

The composite state reconstructor we have designed thus becomes

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{z}} \end{pmatrix} = \left[\begin{array}{c|c} - a + (a + \lambda_1 + \lambda_2) & K \\ \hline - \frac{\lambda_1 \lambda_2}{K} & 0 \end{array}\right] \begin{pmatrix} \hat{x} \\ \hat{z} \end{pmatrix} - \left[\begin{array}{c} (a + \lambda_1 + \lambda_2) \\ \hline - \frac{\lambda_1 \lambda_2}{K} \end{array}\right] \underline{y}$$

$$+ \begin{bmatrix} K \\ 0 \end{bmatrix} \underline{u}_R \quad . \qquad (9)$$

The next step in DAC design is to solve for $\underline{u}_R$. The component $\underline{u}_R$ is to be designed so that $\underline{y}(t)$ follows, quickly and accurately, any admissable servo command $\underline{y}_c(t)$.

Following the suggestion in Reference 3, the servo commands are assumed to be generated by the known dynamical process

$$\left.\begin{aligned} \underline{y}_c &= \underline{G}\ \underline{c} \\ \dot{\underline{c}} &= \underline{E}\ \underline{c} + \underline{\mu} \end{aligned}\right\} \quad (10)$$

where

$\underline{c}$ = "state" of the servo commands

$\underline{G}$, $\underline{E}$ = known, constant matrices

$\underline{\mu}$ = sequence of randomly arriving impulses

Now, in our particular example, $\underline{y}_c$ is piecewise constant. Therefore, just as in the modeling of the disturbance process, we can choose

$$\left.\begin{aligned} \underline{y}_c &= (1)\ c_1 \\ \dot{c}_1 &= \underline{0}\ \underline{c} + \underline{\mu} = \mu \quad . \end{aligned}\right\} \quad (11)$$

A necessary condition for achieving theoretically exact servo tracking is that the "trackability" condition (Reference 4)

$$\underline{G} \equiv \underline{C}\ \underline{\theta}$$

be satisfied for some matrix $\underline{\theta}$. In this case, since

$$\underline{G} = [1],\ \underline{C} = [1]$$

then

$$[1] = [1]\ \theta \text{ " if and only if " } \underline{\theta} = [1] \ .$$

Therefore, one can choose $\underline{\theta} = [1]$ and theoretically exact tracking is possible.

Now, the servo tracking error is

$$\underline{\varepsilon}_y = \underline{y}_c - \underline{y}$$

or

$$\underline{\varepsilon}_y = \underline{G}\,\underline{c} - \underline{C}\,\underline{x} = \underline{C}\,\underline{\theta}\,\underline{c} - \underline{C}\,\underline{x} = \underline{C}\,[\,\underline{\theta}\,\underline{c} - \underline{x}\,] \quad .$$

Let

$$\underline{\varepsilon}_{ss} = \underline{\theta}\,\underline{c} - \underline{x}$$

then

$$\underline{\varepsilon}_y = \underline{C}\,\underline{\varepsilon}_{ss}$$

or

$$\varepsilon_y = (1)\,[\,(1)\,c_1 - x_1\,] = c_1 - x_1 \;. \tag{12}$$

It is desired that $\varepsilon_y \to 0$ rapidly. Thus, we must have

$$c_1 - x_1 \to 0$$

or

$$\varepsilon_{ss} \to 0 \quad .$$

It can be shown (Reference 4) that the differential equation which ε_{ss} obeys is

$$\dot{\underline{\varepsilon}}_{ss} = \underline{A}\,\underline{\varepsilon}_{ss} - \underline{B}\,\underline{u}_R + (\,\underline{\theta}\,\underline{E} - \underline{A}\,\underline{\theta} + \dot{\underline{\theta}}\,)\,\underline{c} - \hat{\underline{B}}\,\underline{z} + \underline{\theta}\,\underline{\mu} \tag{13}$$

where

$$\hat{\underline{B}} = -\,\underline{B}\,\underline{\Gamma} + \underline{F} \quad .$$

This simplifies to

$$\dot{\varepsilon}_{ss} = - a\varepsilon_{ss} - K\underline{u}_R + a\,c_1 + \mu \quad . \tag{14}$$

Now, it is proposed in Reference 4 that $\underline{u}_R$ be chosen as a linear function of $\underline{x}$ and $\underline{c}$ of the form

$$\underline{u}_R = \underline{s}_1\,\underline{x} + \underline{s}_2\,\underline{c} \quad . \tag{15}$$

However, in our design we will instead choose

$$\underline{u}_R = \underline{s}_1\,\underline{\varepsilon}_{ss} + \underline{s}_2\,\underline{c} \quad . \tag{16}$$

In this case

$$u_R = s_1\,\varepsilon_{ss} + s_2\,c_1$$

and

$$\dot{\varepsilon} = - a\varepsilon_{ss} - K(s_1\,\varepsilon_{ss} + s_2\,c_1) + ac_1 + \mu$$

or

$$\dot{\varepsilon} = - (a + K\,s_1)\,\varepsilon_{ss} + (a - K\,s_2)\,c_1 + \mu \quad .$$

In order that $\varepsilon_{ss}(t) \rightarrow 0$ for arbitrary c_1, the term $(a - K\,s_2)c_1$ must be made to be zero. Therefore, $s_2 = a/K$. This leaves

$$\dot{\varepsilon} = - (a + K\,s_1)\,\varepsilon_{ss} + \mu \quad .$$

It is desired that $\varepsilon_{ss}(t) \rightarrow 0$ rapidly between successive incoming impulses of $\mu(t)$. Therefore, we need $(a + K\,s_1)$ to be a large positive number (say in the range 5-10). Therefore, we will set

$$a + K s_1 = \eta \Rightarrow s_1 = \frac{\eta - a}{K} \ .$$

So we now have

$$u_R = s_1 \, \varepsilon_{ss} + s_2 \, c_1 = \left(\frac{\eta - a}{K}\right) \varepsilon_{ss} + \left(\frac{a}{K}\right) c_1$$

$$= \left(\frac{\eta - a}{K}\right) (c_1 - x_1) + \left(\frac{a}{K}\right) c_1$$

$$= \left(\frac{\eta}{K}\right) c_1 + \left(\frac{a - \eta}{K}\right) x_1 \ . \qquad (17)$$

Previously, u_c was found to be (2)

$$u_c = - z$$

Thus,

$$u = u_c + u_R = - z + \left(\frac{a - \eta}{K}\right) x_1 + \left(\frac{\eta}{K}\right) c_1 \qquad \text{(IDEAL)} \ . \qquad (18)$$

From the plant,

$$c_1 = y_c \ , \quad x_1 = y$$

so

$$\hat{u} = - \hat{z} + \left(\frac{a - \eta}{K}\right) y + \left(\frac{\eta}{K}\right) y_c \qquad \text{(PRACTICAL)} \qquad (19)$$

Since y_c and y can be directly measured.

In referring back to the expression for the state reconstructor, let us substitute the derived expression for $\hat{u}$ in place of u.

$$\dot{\hat{x}} = (-a + a + \lambda_1 + \lambda_2)\hat{x} + K\hat{z} - (a + \lambda_1 + \lambda_2)y + Ku$$

$$= (\lambda_1 + \lambda_2)\hat{x} + K\hat{z} - (a + \lambda_1 + \lambda_2)y + K\left[-\hat{z} + \left(\frac{a-\eta}{K}\right)y + \left(\frac{\eta}{K}\right)y_c\right]$$

$$= (\lambda_1 + \lambda_2)\hat{x} - (a + \lambda_1 + \lambda_2 - a + \eta)y + \eta y_c$$

$$= (\lambda_1 + \lambda_2)\hat{x} - (\lambda_1 + \lambda_2 + \eta)y + \eta y_c$$

while $\dot{\hat{z}}$ remains the same,

$$\dot{\hat{z}} = -\frac{\lambda_1 \lambda_2}{K}\hat{x} + \frac{\lambda_1 \lambda_2}{K}y \quad .$$

A diagram of the system with the state reconstructor included is shown in Figure 1.

It is interesting to examine the transfer function interpretation of the DAC servo control scheme we have designed. For this purpose, we will now proceed to find the transfer functions

$$G_c = \frac{y(s)}{y_c(s)} \quad \text{and} \quad G_d = \frac{y(s)}{w(s)} \quad .$$

To find G_c, let disturbance input, $\underline{w} = 0$. Thus, we have

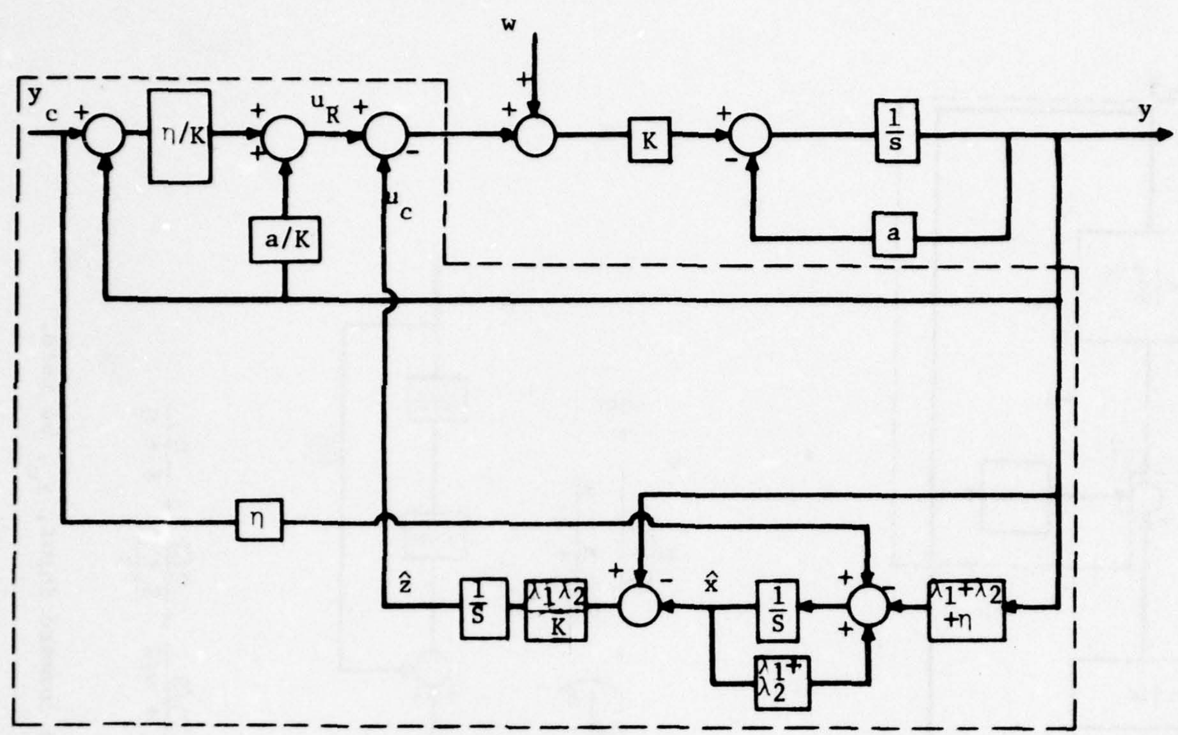
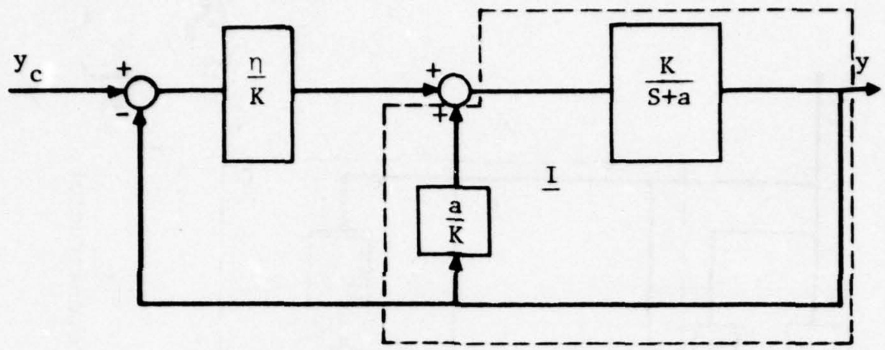


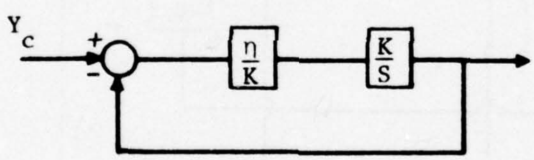
Figure 1. System with DAC and full-dimensional state reconstructor, Case 1A.



or

$$\underline{I}: \quad \frac{\dfrac{K}{S+a}}{1 - \dfrac{a}{K}\left(\dfrac{K}{S+a}\right)} = \frac{\dfrac{K}{S+a}}{\dfrac{S+a-a}{S+a}} = \frac{K}{S}$$

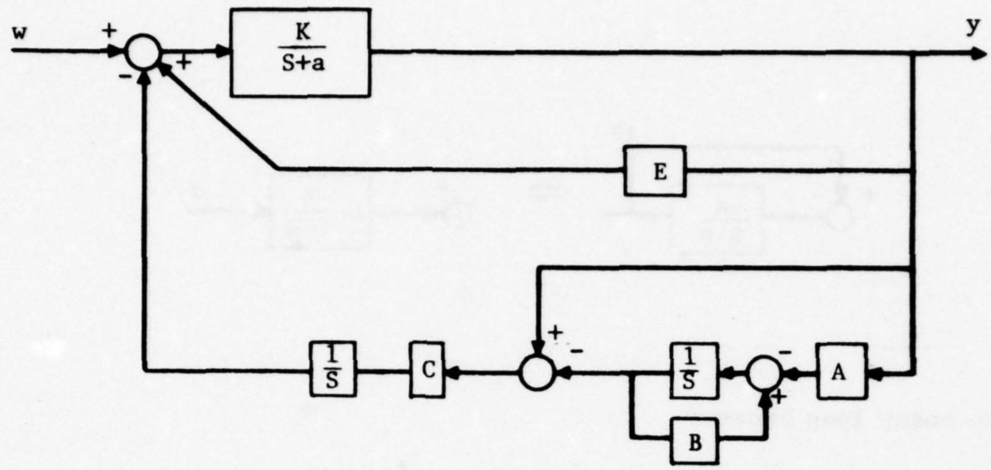
Then



and

$$G_c = \frac{y(s)}{y_c(s)} = \frac{\eta/S}{1+\eta/S} = \frac{\eta/S}{\dfrac{S+\eta}{S}} = \frac{\eta}{S+\eta}$$

To find G_d, let the command input, y_c, be zero.



where

$$A = \lambda_1 + \lambda_2 + \eta$$

$$B = \lambda_1 + \lambda_2$$

$$C = \frac{\lambda_1 \lambda_2}{K}$$

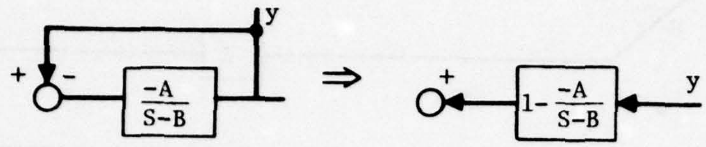
$$E = \frac{a - \eta}{K}$$

Reducing this

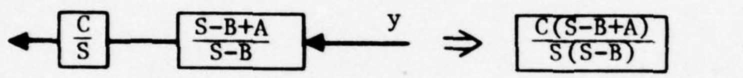
$$\frac{\frac{1}{S}}{1 - B\left(\frac{1}{S}\right)} = \frac{\frac{1}{S}}{\frac{S - B}{S}} = \frac{1}{S - B}$$

$$(-A)\left[\frac{1}{S - B}\right] = \frac{-A}{S - B}$$

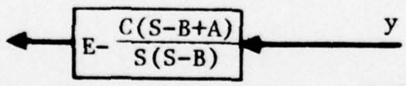
gives



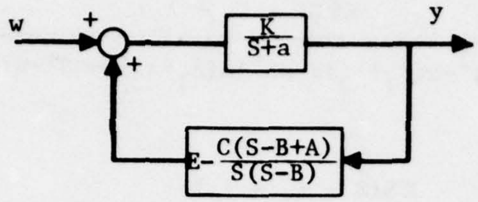
and the outer loop becomes



The two loops can then be combined as



to give



$$\frac{y}{w} = \frac{\dfrac{K}{S+a}}{1 - \left[\dfrac{K}{S+a}\right]\left[E - \dfrac{C(S-B+A)}{S(S-B)}\right]}$$

$$= \frac{\dfrac{K}{S+a}}{1 - \left[\dfrac{K}{S+a}\right]\left[\dfrac{ES(S-B) - C(S-B+A)}{S(S-B)}\right]}$$

$$= \frac{\dfrac{K}{S+a}}{\dfrac{S(S+a)(S-B) - KES(S-B) + KC(S-B+A)}{S(S+a)(S-B)}}$$

$$= \frac{KS(S-B)}{S(S+a)(S-B) - KES(S-B) + KC(S-B+A)}$$

$$= \frac{KS(S-\lambda_1-\lambda_2)}{(S^2+aS)(S-\lambda_1-\lambda_2) - K\left(\dfrac{a-\eta}{K}\right)S(S-\lambda_1-\lambda_2) + K\left(\dfrac{\lambda_1\lambda_2}{K}\right)(S-\lambda_1-\lambda_2+\lambda_1+\lambda_2+\eta)}$$

$$= \frac{KS(S-\lambda_1-\lambda_2)}{S^3-(\lambda_1+\lambda_2)S^2-aS(\lambda_1+\lambda_2)+aS^2-(a-\eta)\left[S^2-(\lambda_1+\lambda_2)S\right]+\lambda_1\lambda_2(S+\eta)}$$

$$= \frac{KS(S - \lambda_1 - \lambda_2)}{S^3-(\lambda_1+\lambda_2-a)S^2-a(\lambda_1+\lambda_2)S-aS^2+a(\lambda_1+\lambda_2)S+\eta S^2-\eta(\lambda_1+\lambda_2)S+\lambda_1\lambda_2 S+\lambda_1\lambda_2\eta}$$

$$= \frac{KS(S - \lambda_1 - \lambda_2)}{S^3 -(\lambda_1+\lambda_2-\eta)S^2 + \left[a-\eta(\lambda_1+\lambda_2) + \lambda_1\lambda_2\right] S + \lambda_1\lambda_2\eta} \quad .$$

From superposition then, the overall closed-loop transfer function for our DAC design is

$$y = \left[\frac{\eta}{S+\eta}\right] y_c + \left[\frac{KS\left[S - (\lambda_1 + \lambda_2)\right]}{S^3-(\lambda_1+\lambda_2-\eta)S^2+\left[a-\eta(\lambda_1+\lambda_2)+\lambda_1\lambda_2\right] S+\lambda_1\lambda_2\eta}\right] w \quad . \tag{20}$$

It was desired to check the dynamic performance of this DAC design by programming it on an analog computer (EAI-221). For a check case, let

$$\begin{aligned} K &= 10 \\ a &= 6 \\ \lambda_1 &= -8 \\ \lambda_2 &= -10 \\ \eta &= 8 \quad . \end{aligned}$$

Thus,

$$G_c = \frac{8}{S + 8}$$

$$G_d = \frac{10S(S + 18)}{S^3 + 26S^2 + 230S + 640}$$

$$= \frac{10S(S + 18)}{(S + 5.365)(S + 10.318 + j3.584)(S + 10.318 - j3.584)}$$

and

$$y = G_c y_c + G_d W \quad .$$

(As a matter of interest, the Bode magnitude and phase plots for G_d are shown in Figures 2 and 3.

An analog computer flow diagram of the above case is shown in Figure 4. Results of the simulation for this design are presented in Figures 7a, 7b, 8 and 9. Also shown is data associated with the design for Case B of this example (Case B will be discussed next).

B. CASE B: DESIGN WITH REDUCED-ORDER COMPOSITE STATE RECONSTRUCTOR

For some practical applications, where reducing equipment requirements and costs is a critical design factor, a lower order state reconstructor may be desirable. A procedure has been developed (Reference 2) for the design of a reduced order composite state reconstructor. The order of the reconstructor designed using this method will be $(n + \rho - m)$ where n is the dimension of the system state vector $\underline{x}$, ρ is the dimension of the disturbance state vector $\underline{z}$, and m is the dimension of the system output vector $\underline{y}$.

From Reference 2, the plant state estimate $\hat{\underline{x}}$ and disturbance state estimate $\hat{\underline{z}}$ are given by the "assembly equations"

$$\hat{\underline{x}} = \left[\underline{C}^T (\underline{C}\, \underline{C}^T)^{-1} - \underline{T}_{12}\, \underline{\Sigma} \right] \underline{y} + \underline{T}_{12}\, \underline{\S} \tag{21}$$

$$\hat{\underline{z}} = \underline{T}_{22} (\underline{\S} - \underline{\Sigma}\, \underline{y}) \tag{22}$$

and $\underline{\S}$ is generated by the dynamical equation (filter prescription)

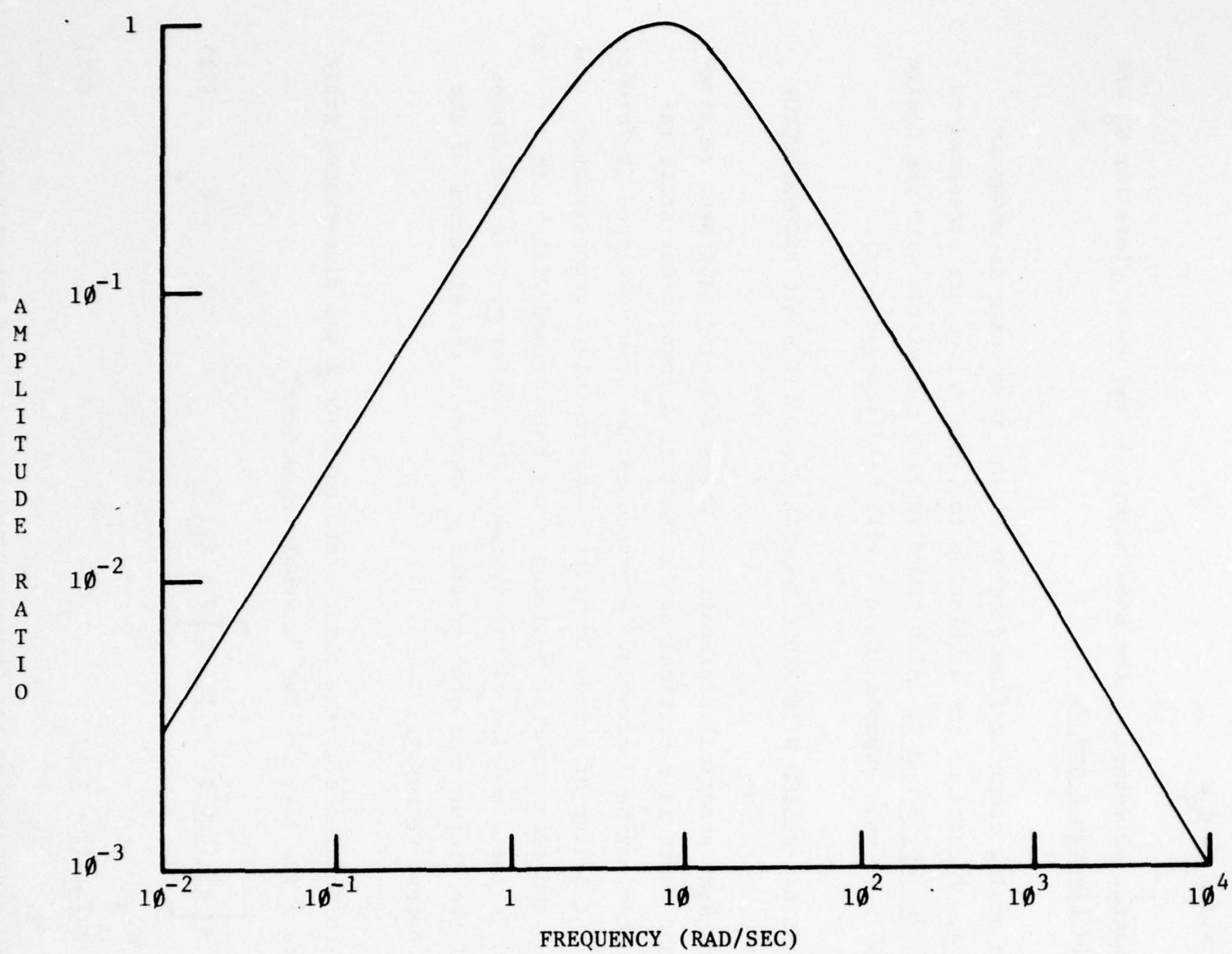


Figure 2. Bode magnitude plot for $G_D = \dfrac{10S\ (S + 18)}{S^3 + 26S^2 + 230S + 640}$

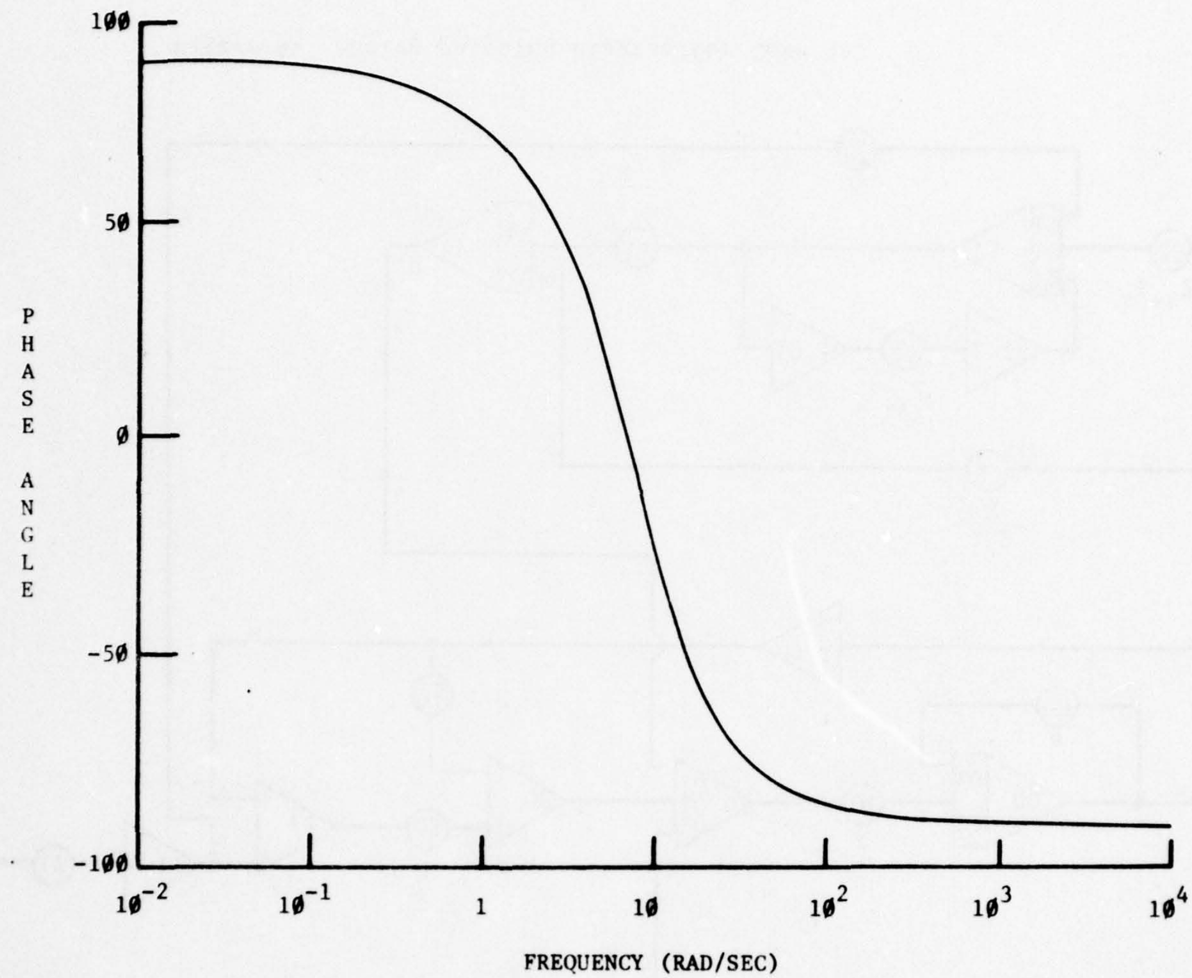


Figure 3. Bode phase plot for $G_D = \dfrac{10S\ (S + 18)}{S^3 + 26S^2 + 230S + 640}$

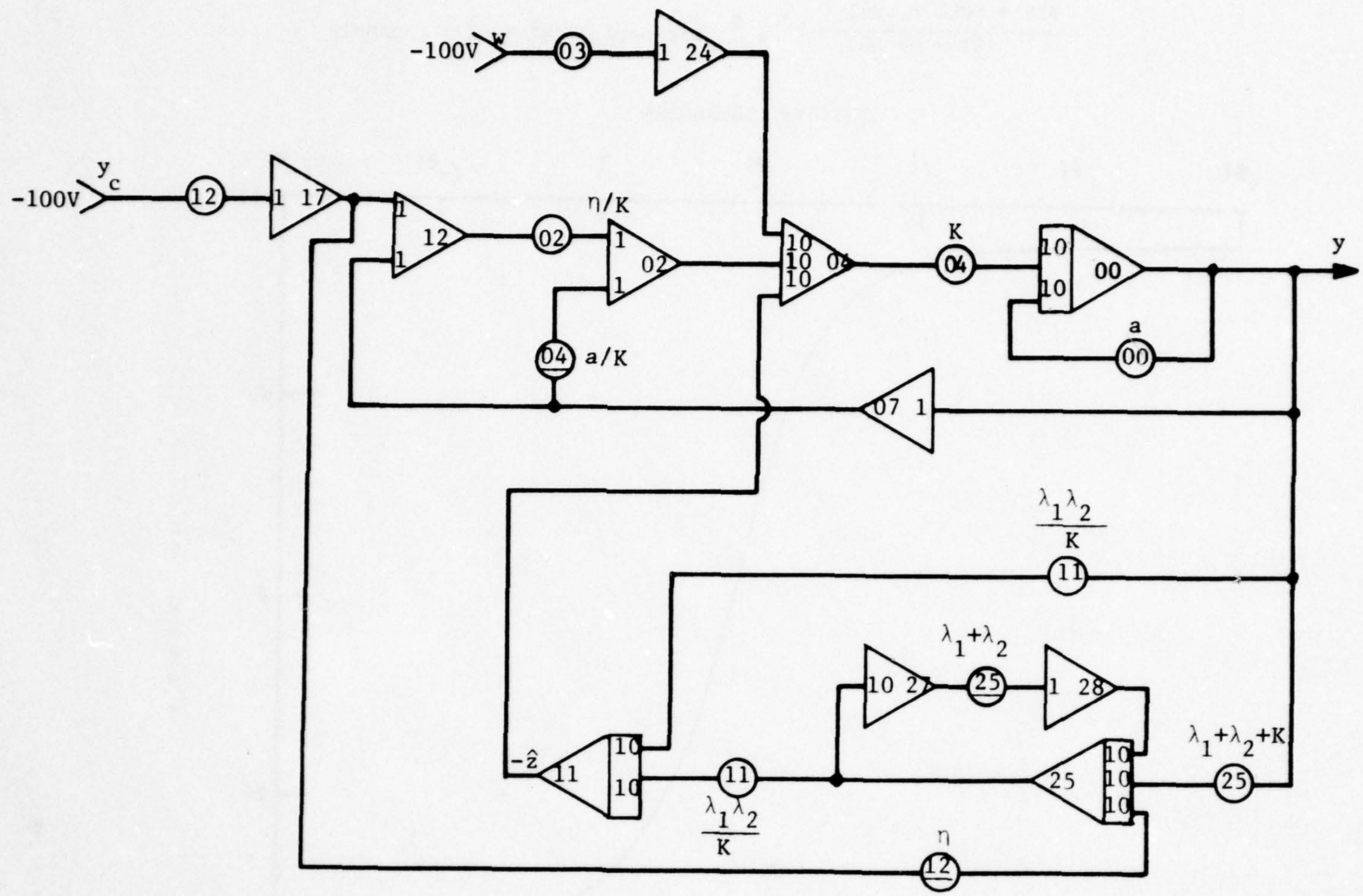


Figure 4. Analog patching diagram for Case 1A.

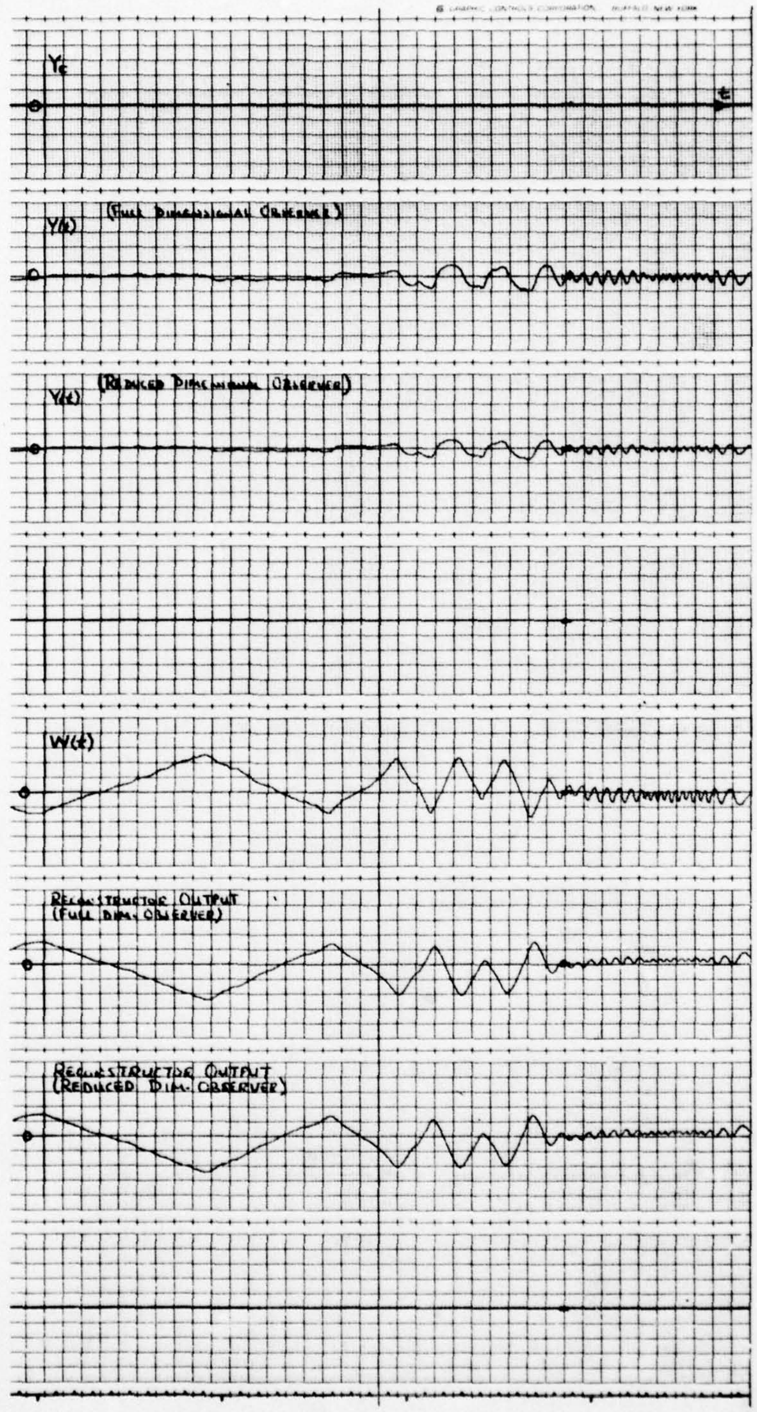


Figure 7a. Output response due to disturbance input.

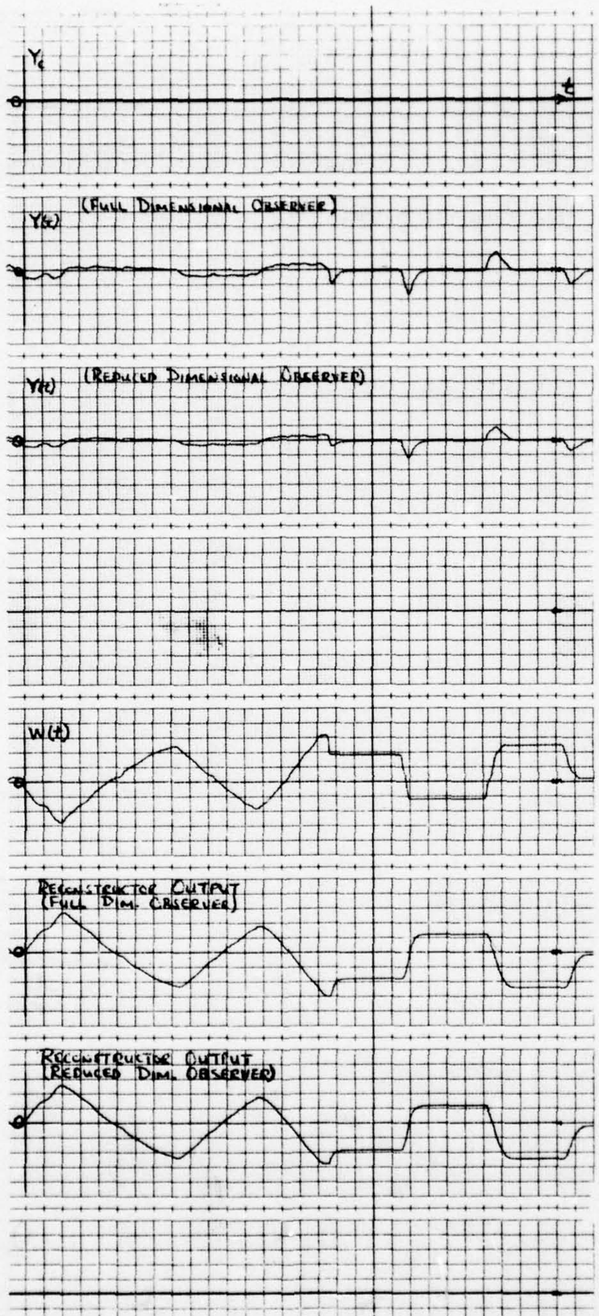


Figure 7b. Output response due to disturbance input.

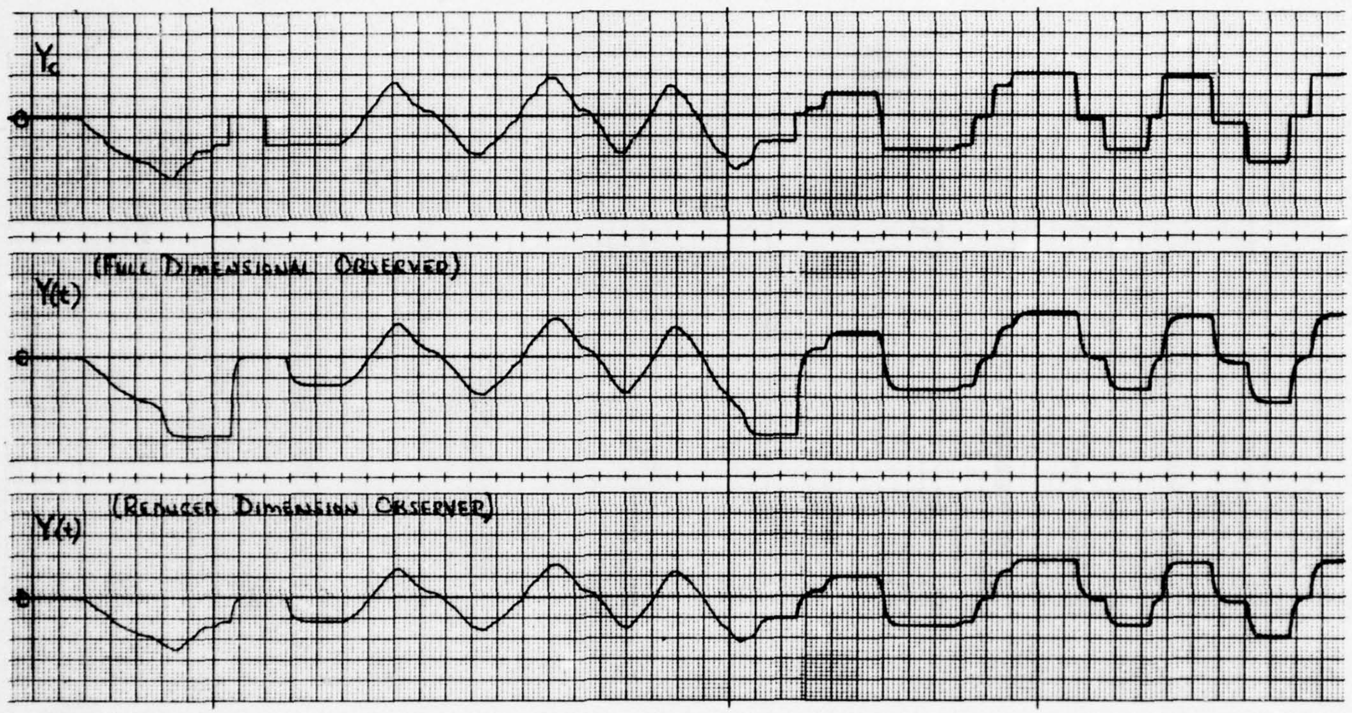


Figure 8. Output response to changes in servo command input.

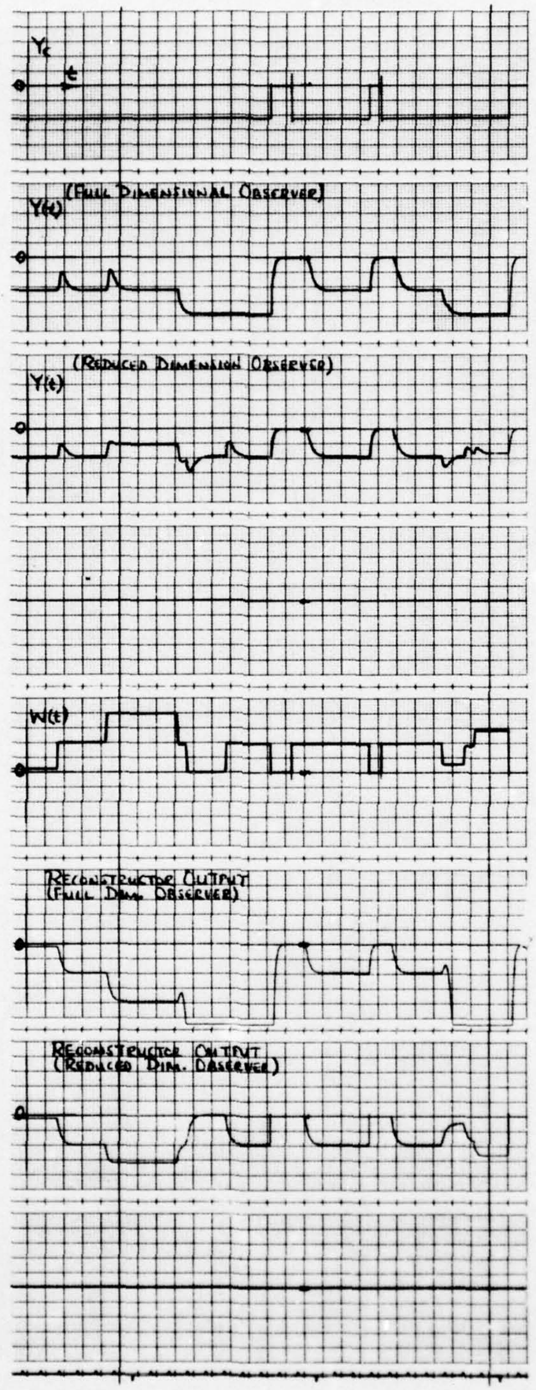


Figure 9. Output response to W(t) for given servo command.

$$\dot{\underline{\S}} = (\underline{\mathcal{D}} + \underline{\Sigma}\, \underline{H})\, \underline{\S} + \underline{\psi}\, \underline{y} + \underline{\Omega}\, \underline{u} \quad . \tag{23}$$

The design of the composite state reconstructor for this case proceeds as follows (Reference 2).

$\underline{T}_{12}$ and $\underline{T}_{22}$ are chosen as any appropriate size matrices satisfying the following conditions

$$[\, \underline{C} \mid \underline{0} \,] \begin{bmatrix} \underline{T}_{12} \\ \underline{T}_{22} \end{bmatrix} \equiv \underline{0} \quad ; \quad \text{RANK} \begin{bmatrix} \underline{T}_{12} \\ \underline{T}_{22} \end{bmatrix} \equiv n + \rho - m \quad .$$

For this problem

$$[\, 1 \quad 0 \,] \begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix} \equiv 0$$

is satisfied for $T_{12} = 0$ and T_{22} any number, say 1. Thus,

$$\underline{T}_{12} = [0] \quad , \quad \underline{T}_{22} = [1]$$

$$\text{RANK} \begin{bmatrix} \underline{T}_{12} \\ \underline{T}_{22} \end{bmatrix} = \text{RANK} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 = n + \rho - m = 1 + 1 - 1 = 1$$

$$\underline{H} = [\, \underline{C} \mid \underline{0} \,] \left(\left[\begin{array}{c|c} \underline{A} & \underline{F}\,\underline{H} \\ \hline \underline{0} & \underline{D} \end{array} \right] \begin{bmatrix} \underline{T}_{12} \\ \underline{T}_{22} \end{bmatrix} - \begin{bmatrix} \dot{\underline{T}}_{12} \\ \dot{\underline{T}}_{22} \end{bmatrix} \right)$$

$$= [\, 1 \; 0 \,] \left(\begin{bmatrix} -a & K \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = [\, 1 \; 0 \,] \begin{bmatrix} K \\ 0 \end{bmatrix} = K$$

$$\underline{\mathcal{D}} = \left[\underline{\overline{T}}_{12} \middle| \underline{\overline{T}}_{22} \right] \left(\left[\begin{array}{c|c} \underline{A} & \underline{F}\ \underline{H} \\ \hline \underline{0} & \underline{D} \end{array} \right] \begin{bmatrix} \underline{T}_{12} \\ \underline{T}_{22} \end{bmatrix} - \begin{bmatrix} \dot{\underline{T}}_{12} \\ \dot{\underline{T}}_{22} \end{bmatrix} \right)$$

where

$$\underline{\overline{T}}_{12} = (\underline{T}_{12}^T \, \underline{T}_{12} + \underline{T}_{22}^T \, \underline{T}_{22})^{-1} \, \underline{T}_{12}^T = (0 + 1)^{-1} \, (0) = 0$$

$$\underline{\overline{T}}_{22} = (\underline{T}_{12}^T \, \underline{T}_{12} + \underline{T}_{22}^T \, \underline{T}_{22})^{-1} \, \underline{T}_{22}^T = (0 + 1)^{-1} \, (1) = 1 \quad .$$

Thus

$$\mathcal{D} = [\ 0 \ 1\ ] \begin{bmatrix} K \\ 0 \end{bmatrix} = 0 \quad .$$

Now

$$\underline{\Sigma} = -\frac{1}{2} \underline{P} \, \underline{H}^T$$

where $\underline{P} > \underline{0}$ is a symmetric matrix satisfying the following equation.

$$\dot{\underline{P}} = \underline{P} \, \underline{\mathcal{D}}^T + \underline{\mathcal{D}} \, \underline{P} - \underline{P} \, \underline{H}^T \, \underline{H} \, \underline{P} + \underline{Q} \quad \text{with } \underline{Q} = \underline{Q}^T > \underline{0}$$

In this example

$$\dot{p}_{11} = p_{11}(0) + (0)p_{11} - p_{11} \, KKp_{11} + q_{11} = p_{11}^2 \, K^2 + q_{11} = 0$$

$$p_{11}^2 = q_{11}/K^2 \quad .$$

Let

$$q_{11} = 1 \Rightarrow p_{11} = 1/K \quad .$$

So

$$\underline{P} = [1/K]$$

and

$$\underline{\Sigma} = -\frac{1}{2}\left(\frac{1}{K}\right)(K) = -\frac{1}{2}$$

$$\underline{\Omega} = (\underline{\overline{T}}_{12} + \underline{\Sigma}\ \underline{C})\ \underline{B} = -\frac{1}{2} K = -K/2$$

$$\underline{C}^{\#} = (\underline{C}\ \underline{C}^{T})^{-1}\ \underline{C} = 1$$

$$\underline{\Psi} = (\underline{\overline{T}}_{12} + \underline{\Sigma}\ \underline{C})(\underline{A}\ \underline{C}^{\#^T} - \underline{\dot{C}}^{\#^T}) - (\underline{D} + \underline{\Sigma}\ \underline{H})\ \underline{\Sigma} + \underline{\dot{\Sigma}}$$

$$= -\frac{1}{2}(-a) - (-K/2)(-\frac{1}{2}) = \frac{a}{2} - \frac{K}{4} \quad .$$

Thus

$$\underline{\dot{\S}} = -\frac{K}{2}\ \underline{\S} + \left(\frac{a}{2} - \frac{K}{4}\right)\underline{y} - \frac{K}{2}\ \underline{u} \quad .$$

Substituting

$$\underline{\hat{u}} = -\hat{z} + \left(\frac{a-\eta}{K}\right) y + \left(\frac{\eta}{K}\right) y_c$$

for u, we obtain

$$\dot{\S} = -\frac{K}{2}\S + \left[\frac{a}{2} - \frac{K}{4} - \left(\frac{a-\eta}{2}\right)\right] y + \frac{K}{2}\hat{z} - \frac{\eta}{2} y_c \quad . \tag{24}$$

Then

$$\hat{x} = [1]y = y \tag{25}$$

$$\hat{z} = \S + \frac{1}{2} y \quad . \tag{26}$$

The estimation errors associated with this reduced-order reconstructor are given by (Reference 2)

$$\underline{x} - \underline{\hat{x}} = \underline{T}_{12}\,\underline{\varepsilon} = \underline{0}$$

$$\underline{z} - \underline{\hat{z}} = \underline{T}_{22}\,\underline{\varepsilon} = \underline{\varepsilon}$$

and

$$\underline{\dot{\varepsilon}} = (\underline{\mathcal{D}} + \underline{\Sigma}\,\underline{\mathcal{H}})\underline{\varepsilon} + \underline{\overline{T}}_{22}\,\underline{\sigma}$$

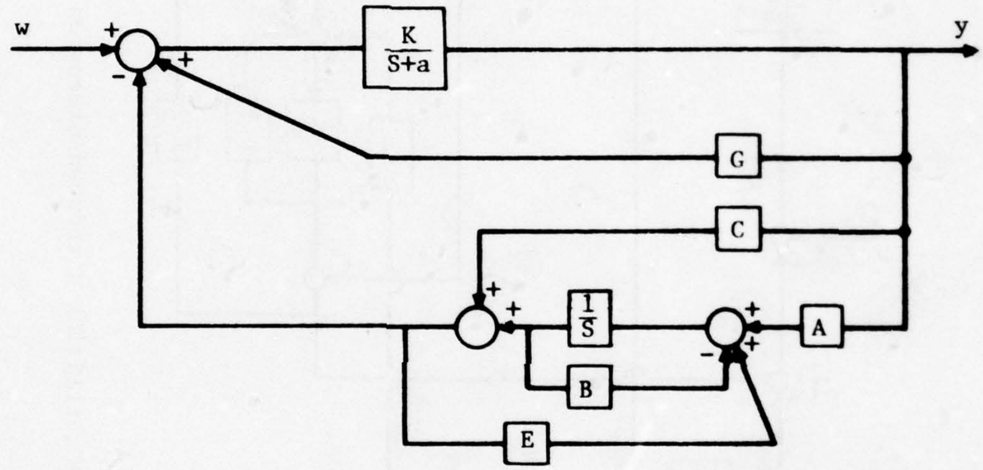
$$= \left[\,0 - \frac{1}{2}K\,\right]\varepsilon + \sigma = -\frac{1}{2}K\varepsilon + \sigma \quad . \tag{27}$$

This constitutes the development of the reduced-order state reconstructor. A diagram of the closed-loop system is shown in Figure 5.

For this case, the transfer function

$$G_c = \frac{y}{y_c}$$

will remain the same as for the full dimension reconstructor case. To find $G_d = \frac{y}{w}$, start with



where

$$A = \frac{a}{2} - \frac{K}{4} - \left(\frac{a - \eta}{2}\right)$$

$$B = K/2$$

$$E = K/2$$

$$C = .5$$

$$G = \frac{a - \eta}{K} \quad .$$

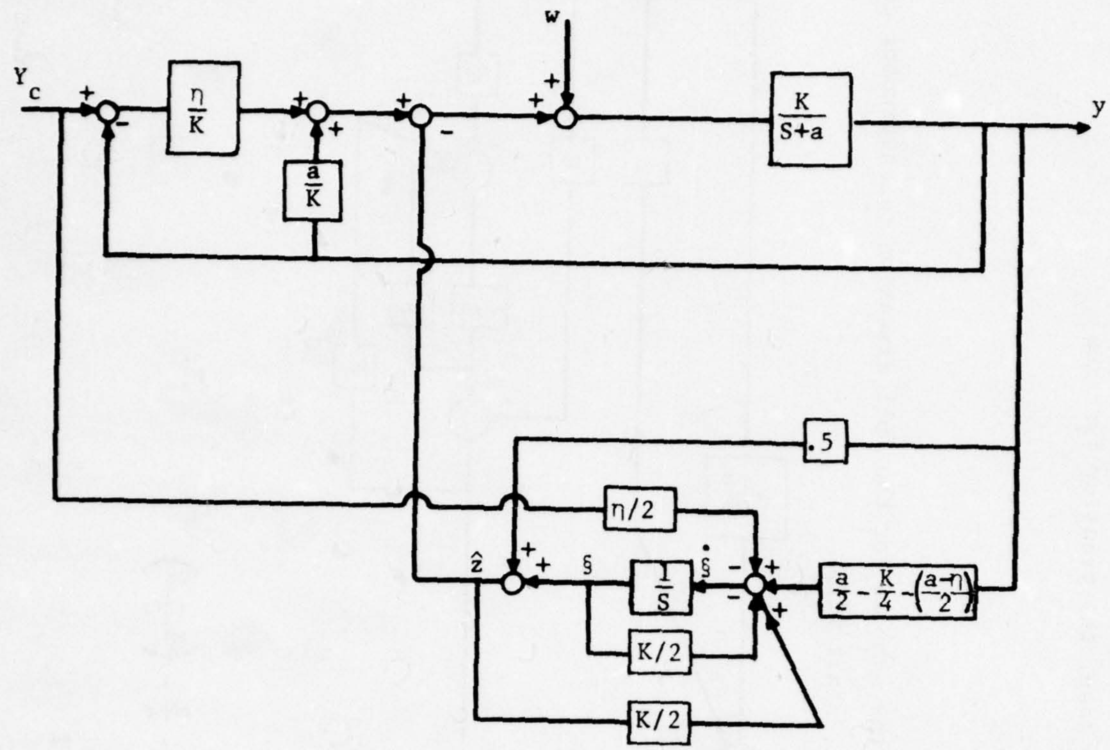
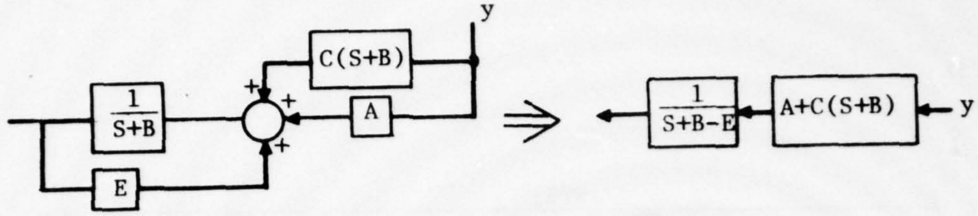
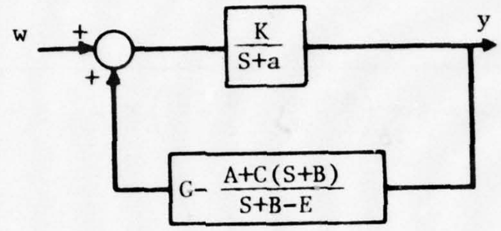


Figure 5. Closed-loop system with DAC utilizing a reduced-dimension state reconstructor, Case 1B.

Reducing the lower section



gives



and

$$\frac{y}{w} = \frac{\dfrac{K}{S + a}}{1 - \left(\dfrac{K}{S + a}\right)\left[\dfrac{G(S + B - E) - A - C(S + B)}{S + B - E}\right]}$$

$$\frac{y}{w} = \frac{\dfrac{K}{S + a}}{\dfrac{(S + a)(S + B - E) - K[G(S + B - E) - A - C(S + B)]}{(S + a)(S + B - E)}}$$

$$= \frac{K(S + B - E)}{(S + a)(S + B - C) - K[G(S + B - E) - A - C(S + B)]} .$$

Now

$$G(S + B - E) = \left(\frac{a - \eta}{K}\right)\left(S + \frac{K}{2} - \frac{K}{2}\right) = \left(\frac{a - \eta}{K}\right) S$$

$$C(S + B) = \frac{1}{2}\left(S + \frac{K}{2}\right) \quad .$$

Thus,

$$\frac{y}{w} = \frac{K(S)}{(S + a)(S) - K\left[\left(\frac{a - \eta}{K}\right) S - \left[\frac{a}{2} - \frac{K}{4} - \left(\frac{a - \eta}{2}\right)\right] - \frac{1}{2}\left(S + \frac{K}{2}\right)\right]}$$

$$= \frac{KS}{S^2 + aS - (a - \eta)S + \left(\frac{a}{2} - \frac{K}{4} - \frac{a}{2} + \frac{\eta}{2}\right) K + \frac{K}{2} S + \frac{K^2}{4}}$$

$$\frac{y}{w} = \frac{KS}{S^2 + \left(a - a + \eta + \frac{K}{2}\right) S + \frac{\eta K}{2}} = \frac{KS}{S^2 + \left(\eta + \frac{K}{2}\right) S + \frac{\eta K}{2}}$$

Utilizing the same values as for the full-dimensional reconstructor

$$\frac{y}{w} = \frac{10S}{S^2 + 13S + 40} = \frac{10S}{(S + 5)(S + 8)}$$

Thus,

$$y = \left(\frac{8}{S + 8}\right) y_c + \left[\frac{10S}{(S + 5)(S + 8)}\right] w \tag{28}$$

This design was also programmed on an analog machine. The flow diagram is shown in Figure 6. Results of actual analog runs are given in Figures 7a, 7b, 8 and 9.

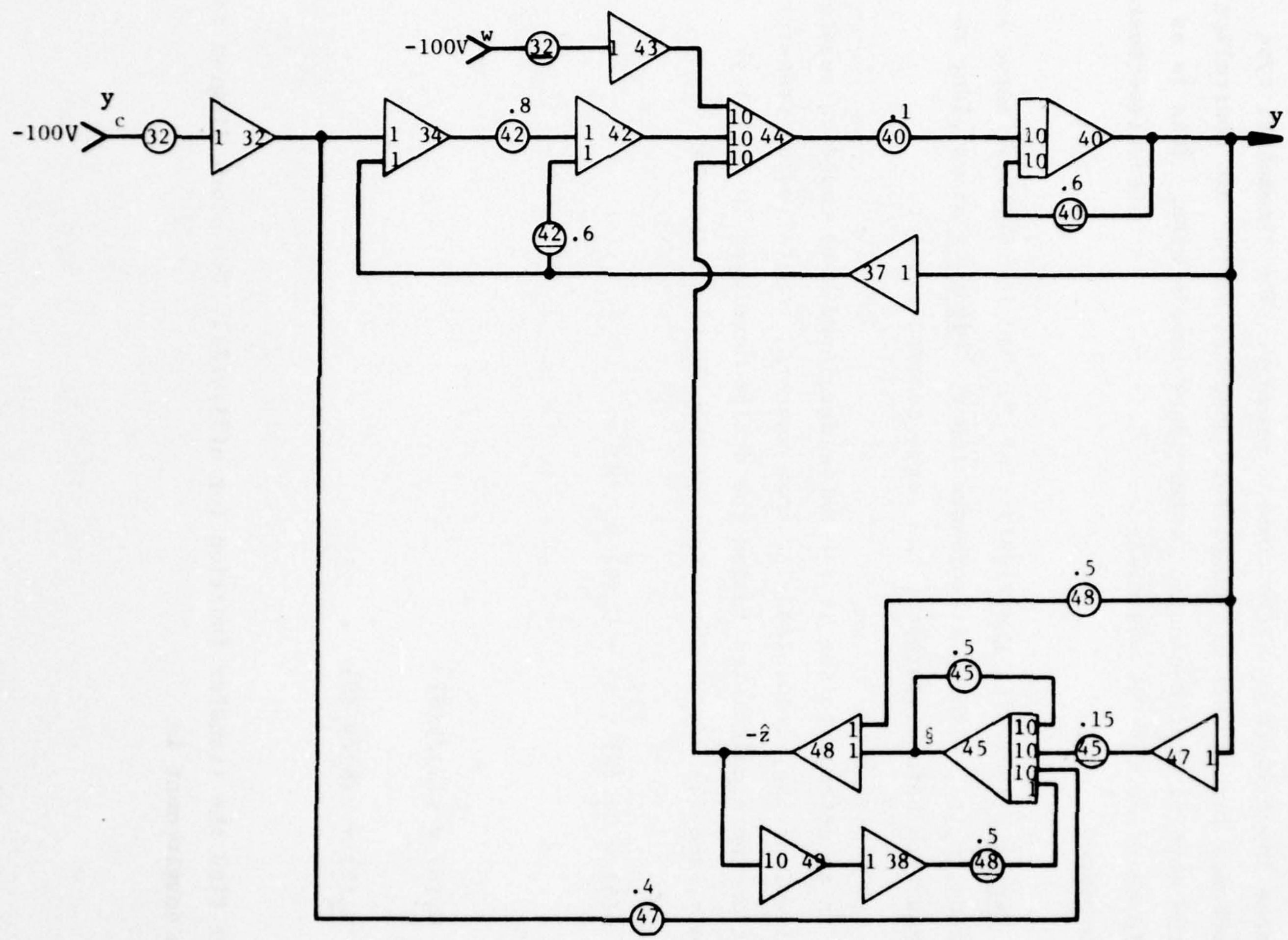


Figure 6. Analog patching diagram for Case 1B.

C. DISCUSSION OF THE ANALOG SIMULATION RESULTS

Figures 7a and 7b show the response of the plant to various disturbance inputs with no servo command present. The sinusoidal type disturbance inputs were not completely absorbed by the DAC controller but the more piecewise-constant (step type) inputs were. This is as it should be since the DAC was designed specifically for piecewise-constant disturbances.

Figure 8 indicates the fidelity of the DAC in following servo input commands, $y_c(t)$, with no disturbance input. Figure 9 shows plant responses with both disturbance and servo command present.

In an attempt to see if the DAC's developed here could be related to classical lag, lead, lead-lag compensators, the following transfer functions were calculated (using the design developed in Part 3.a): (1) u/y, and (2) u/y_c. The output of the DAC would thus be

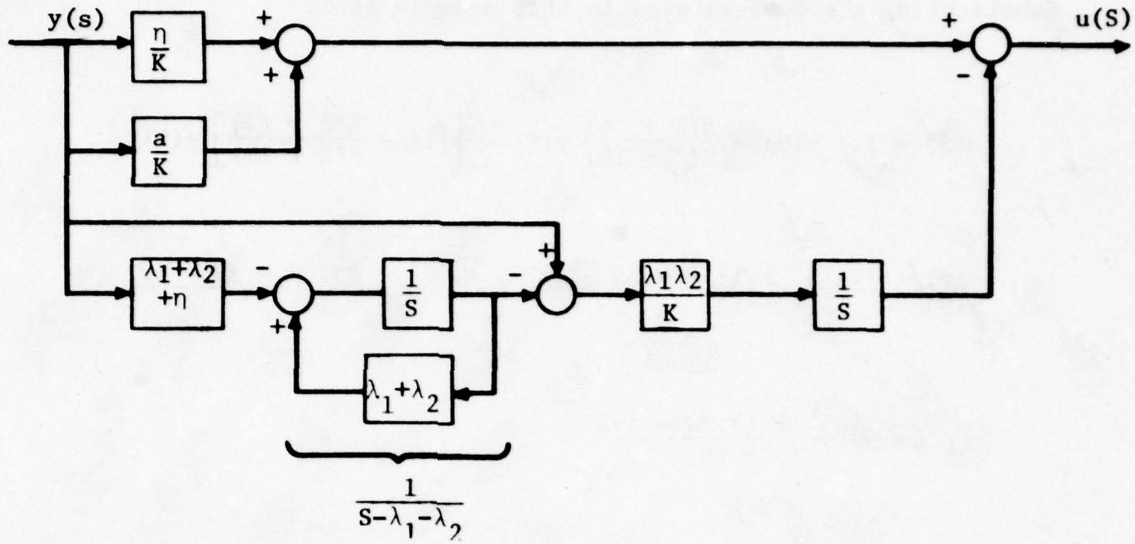
$$u(S) = G_1(S)\, y(S) + G_2(S)\, y_c(S)$$

where

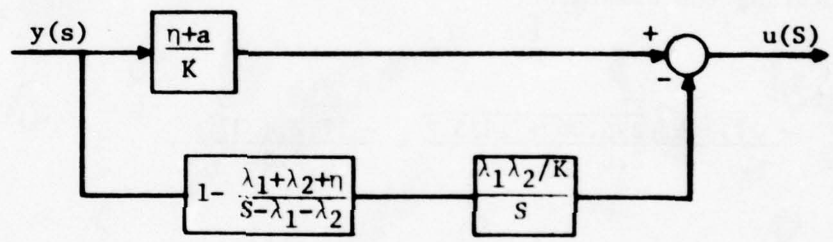
$$G_1(S) = u(S)/y(S)$$

$$G_2(S) = u(S)/y_c(S) \quad .$$

First find the transfer function for u(S)/y(S). The block diagram for this development is



This can be reduced to the following form.



The lower branch can be combined as follows:

$$1 - \frac{\lambda_1 + \lambda_2 + \eta}{S - \lambda_1 - \lambda_2} = \frac{S - \lambda_1 - \lambda_2 - \lambda_1 - \lambda_2 - \eta}{S - \lambda_1 - \lambda_2}$$

$$= \frac{S - 2(\lambda_1 + \lambda_2) - \eta}{S - \lambda_1 - \lambda_2} \quad .$$

Thus,

$$u(S) = \left(\frac{\eta + a}{K}\right) y(S) - \frac{(\lambda_1 \ \lambda_2/K)[S - 2(\lambda_1 + \lambda_2) - \eta]}{S(S - \lambda_1 - \lambda_2)} y(S) \quad .$$

Substituting the numbers used in this example gives

$$u(S) = 1.4\ y(S) - \frac{8(S + 24)}{S(S + 18)}\ y(S) = \left[1.4 - \frac{8(S + 24)}{S(S + 18)}\right] y(S)]$$

$$\frac{u(S)}{y(S)} = \frac{1.4(S^2 + 18S) - 8S - 192}{S(S + 18)} = \frac{1.4S^2 + 25.2S - 8S - 192}{S(S + 18)}$$

$$= \frac{1.4S^2 + 17.2S - 192}{S(S + 18)} \quad .$$

Factoring the numerator:

$$\frac{-17.2 \pm \sqrt{295.84 + 1075.2}}{2.8} = \frac{-17.2 \pm 37}{2.8} \quad .$$

So,

$$S_1 = -27.1$$

$$S_2 = 9.9 \qquad (29)$$

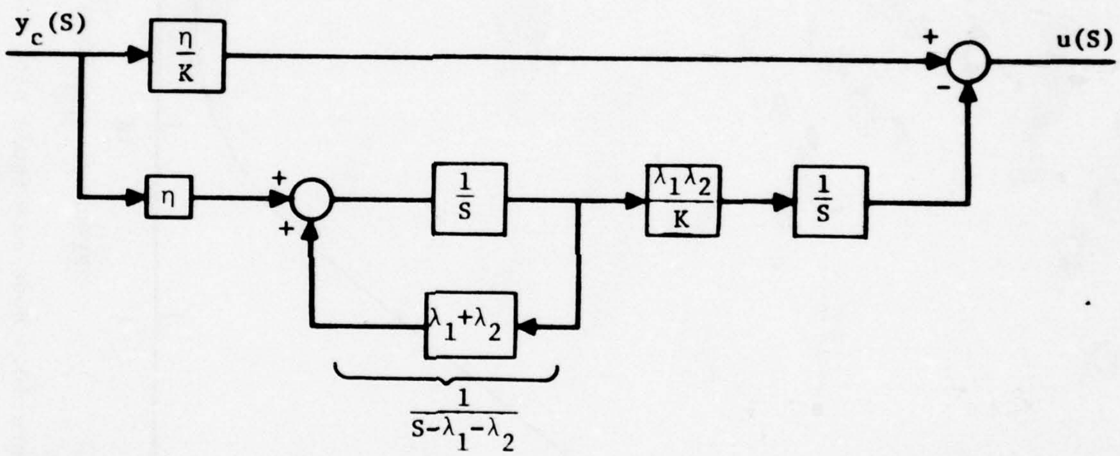
and

$$\frac{u(S)}{y(S)} = \frac{(S + 27.1)(S - 9.9)}{S(S + 18)} \quad .$$

This is a non-minimum phase transfer function and does not conform to the models of any of the previously mentioned classical compensators. The Bode magnitude sketch of u(S)/y(S) acts like two classical lag compensators in series.

Figures 10 and 11 show the Bode plots for this transfer function. The break frequencies at $\omega = 18$ and $\omega = 27.1$ are too close to give noticeable effects on these plots.

Next, find the transfer function for $u(S)/y_c(S)$. The block diagram for this development is



The lower branch can be reduced to

$$\left[\frac{\eta}{S - \lambda_1 - \lambda_2}\right]\left[\frac{\lambda_1 \lambda_2/K}{S}\right] = \frac{\lambda_1 \lambda_2 \eta/K}{S(S - \lambda_1 - \lambda_2)}$$

So,

$$u(S) = \frac{\eta}{K} y_c(S) - \left[\frac{\lambda_1 \lambda_2 \eta/K}{S(S - \lambda_1 - \lambda_2)}\right] y_c(S)$$

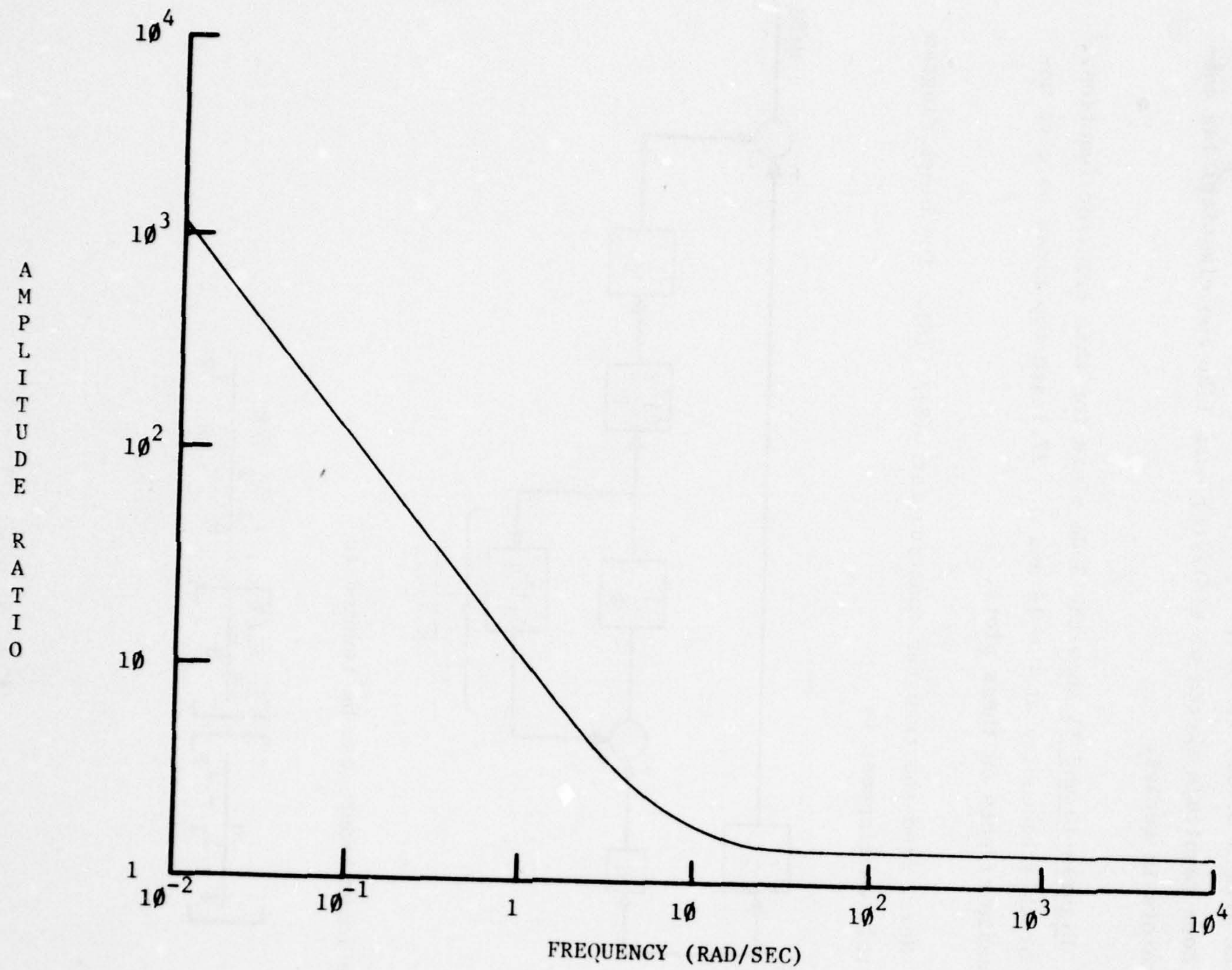


Figure 10. Bode magnitude plot for u(S)/Y(S).

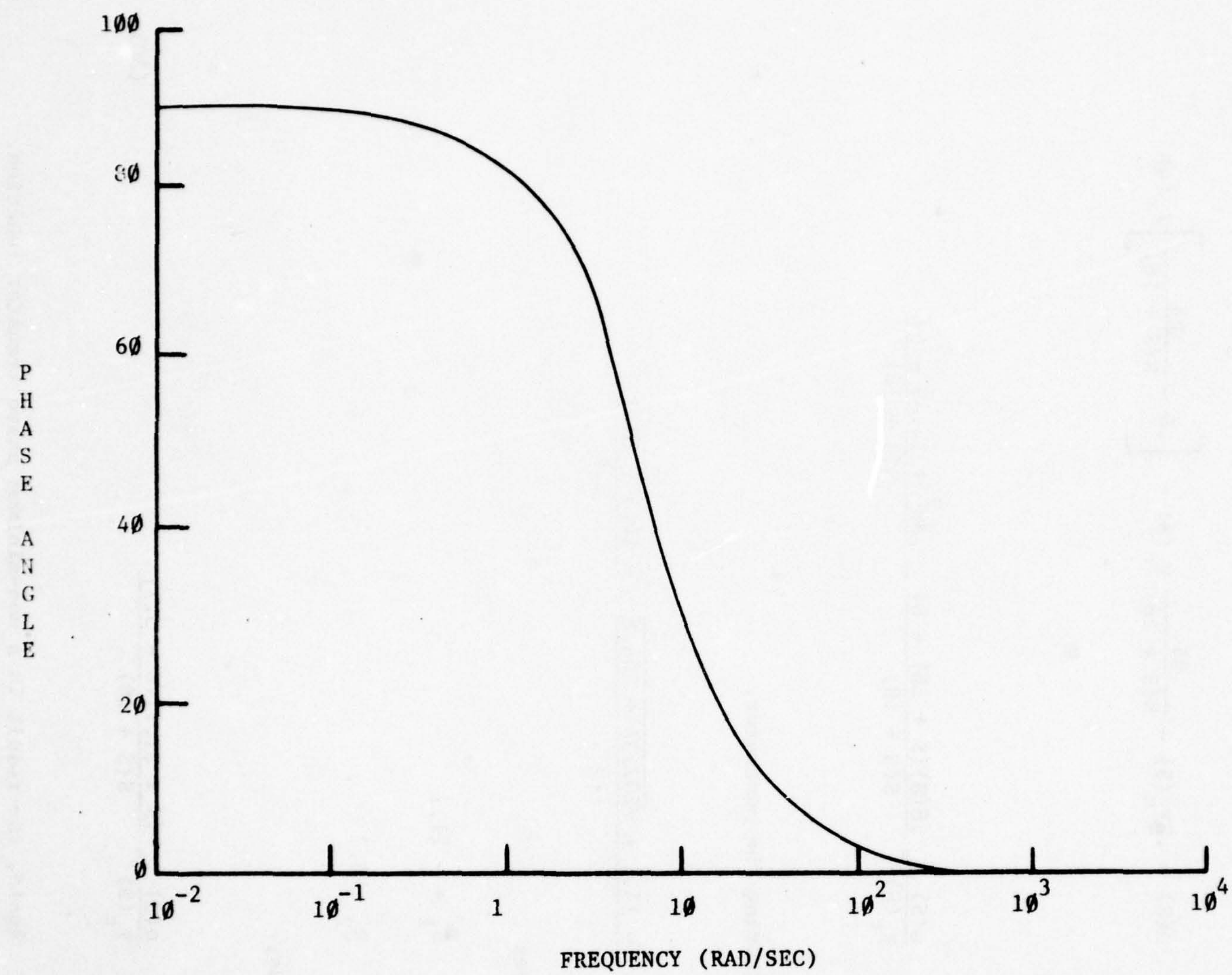


Figure 11. Bode phase plot for u(S)/Y(S).

Substituting values for this case gives

$$u(S) = .8y_c(S) - \frac{64}{S(S + 18)} y_c(S) = \left[.8 - \frac{64}{S(S + 18)}\right] y_c(s)$$

and

$$\frac{u(S)}{y_c(S)} = \frac{.8(S)(S + 18) - 64}{S(S + 18)} = \frac{.8S^2 + 14.4S - 64}{S(S + 18)} \quad .$$

Factoring the numerator,

$$\frac{- 14.4 \pm \sqrt{207.36 + 204.8}}{1.6} = \frac{- 14.4 \pm 20.3}{1.6}$$

gives

$$S_1 = - 21.7$$

$$S_2 = 3.7 \quad .$$

Thus,

$$\frac{u(S)}{y_c(S)} = \frac{(S + 21.7)(S - 3.7)}{S(S + 18)} \quad . \tag{30}$$

Again, the result is a non-minimum phase transfer function. Figures 12 and 13 give the Bode plots for this transfer function.

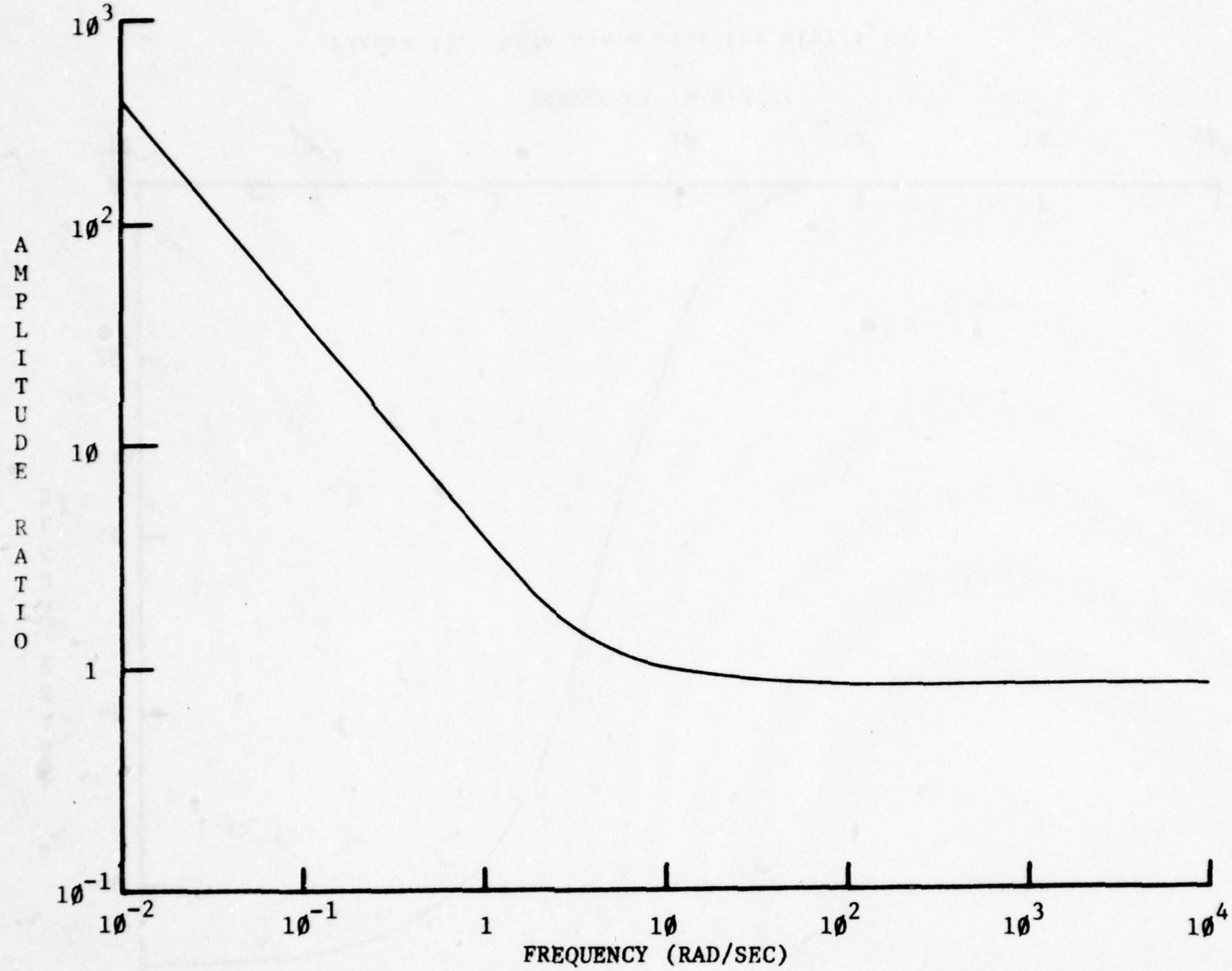


Figure 12. Bode magnitude plot for $u(S)/Y_c(S)$.

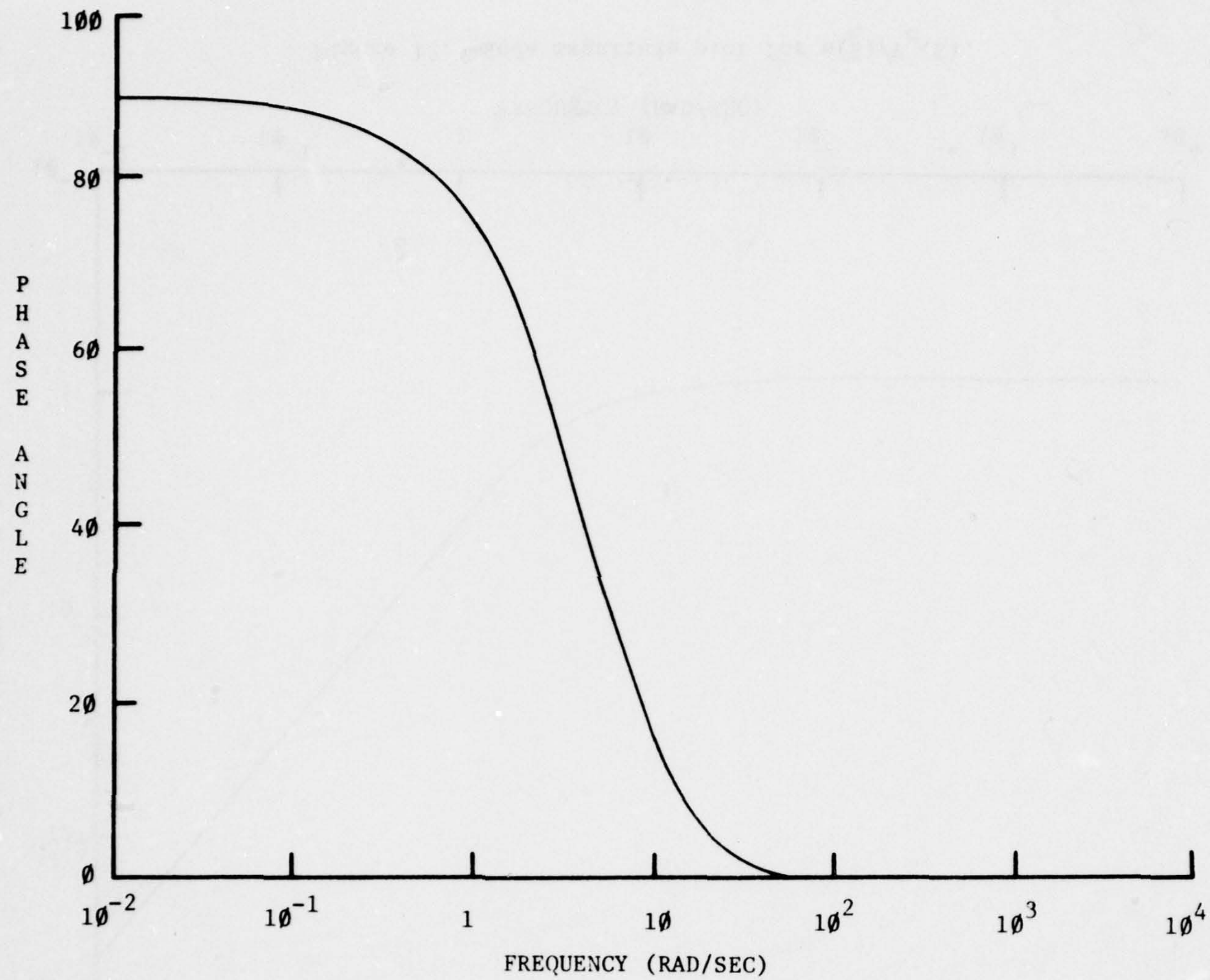


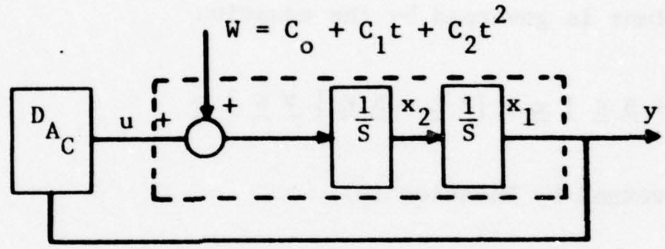
Figure 13. Bode phase plot for $u(S)/Y_c(S)$.

4. EXAMPLE DESIGN PROBLEM NO. 2

For the next example design, an output regulator problem is considered for a second-order plant subject to a "step-ramp-acceleration" input disturbance $w(t) = C_0 + C_1 t + C_2 t^2$. The objective here is to design a DAC, using a reduced order state reconstructor, which will counteract $w(t)$ and make the plant output $y(t)$ rapidly settle to zero while obeying the closed-loop characteristic equation

$$\ddot{y} + \alpha_2 \dot{y} + \alpha_1 y = 0$$

where (α_1, α_2) are given. A diagram of the plant and controller is shown below.



To begin, we have

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u + w .$$

Thus, for the general form

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} + \underline{F}\,\underline{w}$$

we obtain

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \underline{u} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \underline{w} \tag{31}$$

and, from

$$\underline{y} = \underline{C}\ \underline{x}$$

we have

$$\underline{y} = (1\ 0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ . \tag{32}$$

Also, $\underline{L} \equiv \underline{0}$, $\underline{M} \equiv \underline{0}$ since the disturbance is not dependent on the plant state.

Now, it can be shown (Reference 1) that the state $\underline{x}(t)$ of the closed-loop plant is governed by the equation

$$\dot{\underline{x}} = [\ \underline{A} + \underline{B}\ \underline{K}\ ]\ \underline{x} + [\underline{F}\ \underline{L} - \underline{B}\ \underline{K}\ |\ \underline{F}\ \underline{H}\ ]\ \underline{\varepsilon} \tag{33}$$

where $\underline{\varepsilon}$ is governed by Equation (4).

For this example then,

$$\dot{\underline{x}} = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left\{ \begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \Bigg| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \right\} \underline{\varepsilon}$$

$$= \begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -k_1 & -k_2 & 1 & 0 & 0 \end{bmatrix} \underline{\varepsilon} \ . \tag{34}$$

Solving for the characteristic polynomial:

$$[(\underline{A} + \underline{BK}) - \lambda \underline{I}] = \begin{vmatrix} -\lambda & 1 \\ k_1 & (k_2 - \lambda) \end{vmatrix} = \lambda^2 - k_2\ \lambda - k_1 \ .$$

Comparing this to the closed-loop characteristic equation

$$S^2 + \alpha_2 S + \alpha_1$$

we see that we must have

$$k_2 = -\alpha_2$$

$$k_1 = -\alpha_1$$

therefore

$$\underline{u}_R = \underline{K}\,\underline{x} = \begin{bmatrix} -\alpha_1 & -\alpha_2 \end{bmatrix} \underline{x} \tag{35}$$

and since $\underline{x}(t)$ is assumed unmeasurable, for practical implementation we must have

$$\underline{\hat{u}}_R = \begin{bmatrix} -\alpha_1 & -\alpha_2 \end{bmatrix} \underline{\hat{x}} \quad . \tag{36}$$

To describe the disturbance in state-variable form, we must first find a differential equation which is satisfied by

$$w(t) = C_0 + C_1 t + C_2 t^2 \quad .$$

Taking the laplace transform,

$$w(S) = \frac{C_0}{S} + \frac{C_1}{S^2} + \frac{2C_2}{S^3} = \frac{C_0 S^2 + C_1 S + 2C_2}{S^3} \quad .$$

Thus, the characteristic polynomial associated with w(t) is

$$\lambda^3 = 0 \quad .$$

Therefore, we will choose the matrices in the model

$$\underline{w} = \underline{H}\ \underline{z}$$

$$\underline{\dot{z}} = \underline{D}\ \underline{z} + \underline{\sigma}$$

such that $\underline{\dot{z}} = \underline{D}\ \underline{z}$ has the characteristic polynomial $\lambda^3 = 0$ and $\underline{H}\ \underline{z}$ has the general form

$$w = C_0 + C_1 t + C_2 t^2 \quad .$$

For this purpose, let

$$w = \underbrace{(1\ 0\ 0)}_{H} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

and

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \underbrace{\begin{bmatrix} -\beta_3 & 1 & 0 \\ -\beta_2 & 0 & 1 \\ -\beta_1 & 0 & 0 \end{bmatrix}}_{\underline{D}} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \underline{\sigma} \quad .$$

Find

$$\det|\underline{D} - \lambda\underline{I}| = \begin{bmatrix} -\beta_3 - \lambda & 1 & 0 \\ -\beta_2 & -\lambda & 1 \\ -\beta_1 & 0 & -\lambda \end{bmatrix} = -(\beta_3 + \lambda)(\lambda^2) - \beta_1 - \beta_2\lambda = 0$$

or

$$\lambda^3 + \beta_3 \lambda^2 + \beta_2 \lambda + \beta_1 = 0 .$$

Comparing with $\lambda^3 = 0 \Rightarrow \beta_1 = \beta_2 = \beta_3 = 0$, thus

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \underline{\sigma} \quad . \tag{37}$$

Check for existence of $\underline{u}_c$ (Reference 1):

$$\underline{F} \equiv \underline{B}\ \underline{\Gamma}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1) \Rightarrow \text{ satisfied for } \Gamma \equiv (1) \quad .$$

Therefore, let

$$\underline{u}_c = - \underline{w} = - (1\ 0\ 0) \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \tag{38}$$

Thus,

$$\hat{u}_c = - \hat{z}_1 \ . \tag{39}$$

Now, develop a reduced-dimension observer to generate $\hat{x}$, $\hat{z}$.

From input model, $\rho = 3$.
From output $m = 1$.
From state model, $n = 2$.
So $(n + \rho - m) = 4$.

Now,

$$\underline{\hat{x}} = [\ \underline{C}^T(\underline{C}\ \underline{C}^T)^{-1} - \underline{T}_{12}\ \underline{\Sigma}\ ]\ \underline{y} + \underline{T}_{12}\ \underline{\S} \tag{40}$$

$$\underline{\hat{z}} = \underline{T}_{22}\ (\underline{\S} - \underline{\Sigma}\ \underline{y}) \tag{41}$$

where

$$\underline{\dot{\S}} = (\underline{\mathcal{D}} + \underline{\Sigma}\ \underline{\mathcal{H}})\underline{\S} + \underline{\psi}\ \underline{y} + \underline{\Omega}\ \underline{u} \tag{42}$$

$\underline{T}_{12}$ is an $n \times (n + \rho - m) = 2 \times 4$ matrix

$\underline{T}_{22}$ is a $\rho \times (n + \rho - m) = 3 \times 4$ matrix

where

$$[\ \underline{C} \mid \underline{0}\ ] \begin{bmatrix} \underline{T}_{12} \\ \underline{T}_{22} \end{bmatrix} \equiv \underline{0} \quad \text{and} \quad \text{rank} \begin{bmatrix} \underline{T}_{12} \\ \underline{T}_{22} \end{bmatrix} \equiv n + \rho - m = 4 \, .$$

Substituting, we have

$$[\ 1\ 0 \mid 0\ 0\ 0\ ] \underbrace{\begin{bmatrix} t_{12_{11}} & t_{12_{12}} & t_{12_{13}} & t_{12_{14}} \\ t_{12_{21}} & t_{12_{22}} & t_{12_{21}} & t_{12_{24}} \\ t_{22_{11}} & t_{22_{12}} & t_{22_{13}} & t_{22_{14}} \\ t_{22_{21}} & t_{22_{22}} & t_{22_{23}} & t_{22_{24}} \\ t_{22_{31}} & t_{22_{32}} & t_{22_{33}} & t_{22_{34}} \end{bmatrix}} \equiv \underline{0} \, .$$

Let

$$\underline{T}_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \underline{T}_{22} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \, .$$

Then

$$[\ 1\ 0 \mid 0\ 0\ 0\ ] \begin{bmatrix} \underline{T}_{12} \\ \hline \underline{T}_{22} \end{bmatrix} \equiv \underline{0}$$

and

$$\text{rank} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = n + \rho - m = 4 \ .$$

Continuing

$$\bar{\underline{T}}_{12} = \left(\underline{T}_{12}^T \, \underline{T}_{12} + \underline{T}_{22}^T \, \underline{T}_{22}\right)^{-1} \underline{T}_{12}^T$$

$$\underline{T}_{12}^T \, \underline{T}_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{T}_{22}^T \, \underline{T}_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Thus

$$\underline{T}_{12}^T \, \underline{T}_{12} + \underline{T}_{22}^T \, \underline{T}_{22} = \begin{bmatrix} 1 & & & \bigcirc \\ & 1 & & \\ & & 1 & \\ \bigcirc & & & 1 \end{bmatrix} = \underline{I}$$

so

$$\left(\underline{T}_{12}^T \, \underline{T}_{12} + \underline{T}_{22}^T \, \underline{T}_{22}\right)^{-1} = \underline{I}^{-1} = \underline{I}$$

and

$$\overline{\underline{T}}_{12} = \underline{T}_{12}^{T} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

Now,

$$\overline{\underline{T}}_{22} = \left(\underline{T}_{12}^{T}\ \underline{T}_{12} + \underline{T}_{22}^{T}\ \underline{T}_{22}\right)^{-1} \underline{T}_{22}^{T} = \underline{I}\ \underline{T}_{22}^{T} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Thus

$$\underline{D} = \left[\overline{\underline{T}}_{12} \mid \overline{\underline{T}}_{22}\right] \left(\left[\begin{array}{c|c} \underline{A} & \underline{F}\ \underline{H} \\ \hline \underline{0} & \underline{D} \end{array}\right] \left[\begin{array}{c} \underline{T}_{12} \\ \hline \underline{T}_{22} \end{array}\right] \right)$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \left(\left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{\underline{H}} = [\ \underline{C} \mid \underline{0}\ ] \left(\left[\begin{array}{c|c} \underline{A} & \underline{F}\ \underline{H} \\ \hline \underline{0} & \underline{D} \end{array} \right] \left[\begin{array}{c} \underline{T}_{12} \\ \hline \underline{T}_{22} \end{array} \right] \right)$$

$$= [\ 1\ 0\ 0\ 0\ 0\ ] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = [\ 1\ 0\ 0\ 0\ ]\ .$$

Now, from the error dynamics (Reference 2)

$$\underline{x} - \underline{\hat{x}} = \underline{T}_{12}\ \underline{\varepsilon}$$

$$\underline{z} - \underline{\hat{z}} = \underline{T}_{22}\ \underline{\varepsilon}$$

and

$$\underline{\dot{\varepsilon}} = (\mathcal{\underline{D}} + \underline{\Sigma}\ \mathcal{\underline{H}})\ \underline{\varepsilon} + \underline{\overline{T}}_{22}\ \underline{\sigma} \quad .$$

To have $\underline{\varepsilon}$ approach zero rapidly, it is desired that $(\mathcal{\underline{D}} + \underline{\Sigma}\ \mathcal{\underline{H}})$ be "large" negative values. Using the relations for $\mathcal{\underline{D}}$ and $\mathcal{\underline{H}}$ we have

$$\underline{\dot{\varepsilon}} = \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \Sigma_{11} \\ \Sigma_{21} \\ \Sigma_{31} \\ \Sigma_{41} \end{bmatrix} [\ 1\ 0\ 0\ 0\ ] \right) \underline{\varepsilon}$$

$$= \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \Sigma_{11} & 0 & 0 & 0 \\ \Sigma_{21} & 0 & 0 & 0 \\ \Sigma_{31} & 0 & 0 & 0 \\ \Sigma_{41} & 0 & 0 & 0 \end{bmatrix} \right) \underline{\varepsilon} \quad .$$

Thus

$$\begin{bmatrix} \dot{\varepsilon}_1 \\ \dot{\varepsilon}_2 \\ \dot{\varepsilon}_3 \\ \dot{\varepsilon}_4 \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & 1 & 0 & 0 \\ \Sigma_{21} & 0 & 1 & 0 \\ \Sigma_{31} & 0 & 0 & 1 \\ \Sigma_{41} & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{pmatrix} \quad .$$

Now, find the roots of this characteristic matrix.

$$\left| (\underline{D} + \underline{\Sigma}\ \underline{H}) - \lambda\ \underline{I} \right| = \begin{vmatrix} (\Sigma_{11} - \lambda) & 1 & 0 & 0 \\ \Sigma_{21} & -\lambda & 1 & 0 \\ \Sigma_{31} & 0 & -\lambda & 1 \\ \Sigma_{41} & 0 & 0 & -\lambda \end{vmatrix}$$

$$= (\Sigma_{11} - \lambda) \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} - (1) \begin{vmatrix} \Sigma_{21} & 1 & 0 \\ \Sigma_{31} & -\lambda & 1 \\ \Sigma_{41} & 0 & -\lambda \end{vmatrix} = 0$$

so

$$0 = (\Sigma_{11} - \lambda)(-\lambda^3) - (1)(\Sigma_{21}\ \lambda^2 + \Sigma_{41} + \Sigma_{31}\ \lambda)$$

$$= \lambda^4 - \Sigma_{11}\,\lambda^3 - \Sigma_{21}\,\lambda^2 - \Sigma_{31}\,\lambda - \Sigma_{41} \quad .$$

We know that

$$0 = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$$

or

$$0 = \lambda^4 - (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)\lambda^3 + (\lambda_1\lambda_2 + \lambda_3\lambda_4 + \lambda_1\lambda_3 + \lambda_1\lambda_4$$

$$+ \lambda_2\lambda_3 + \lambda_2\lambda_4)\lambda^2 - (\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4)\lambda +$$

$$\lambda_1\lambda_2\lambda_3\lambda_4 \quad .$$

Equating coefficients

$$- \Sigma_{11} = - (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)$$

$$- \Sigma_{21} = \lambda_1\lambda_2 + \lambda_3\lambda_4 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4$$

$$- \Sigma_{31} = - (\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4)$$

$$- \Sigma_{41} = \lambda_1\lambda_2\lambda_3\lambda_4 \quad .$$

It is desired that the roots have "large" negative values. Hence, let

$$\lambda_1 = - 6$$

$$\lambda_2 = - 8$$

$$\lambda_3 = -10 + j5$$

$$\lambda_4 = -10 - j5 \quad .$$

In this case

$$\Sigma_{11} = -6 - 8 - 10 - 10 = -34$$

$$-\Sigma_{21} = 48 + 125 + (60 - 6j5) + (60 + 6j5) + (80 - 8j5)$$

$$+ (80 + 8j5) = 453$$

$$\Sigma_{31} = 48\ (-10 + j5) + 48\ (-10 - j5) + (-6)(-10 + j5)(-10 - j5)$$

$$+ (-8)(-10 + j5)(-10 - j5)$$

$$= -480 - 480 - 750 - 1000 = -2710$$

$$-\Sigma_{41} = 48\ (125) = 6000 \quad .$$

So

$$\dot{\underline{\varepsilon}} = \begin{bmatrix} -34 & 1 & 0 & 0 \\ -453 & 0 & 1 & 0 \\ -2710 & 0 & 0 & 1 \\ -6000 & 0 & 0 & 0 \end{bmatrix} \underline{\varepsilon} + \underline{\overline{T}}_{22}\ \underline{\sigma} \quad . \tag{43}$$

Proceeding with the remainder of the problem

$$\underline{\Omega} = (\underline{\overline{T}}_{12} + \underline{\Sigma}\ \underline{C})\ \underline{B}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -34 \\ -453 \\ -2710 \\ -6000 \end{bmatrix} [\ 1\ 0\ ] \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -34 & 0 \\ -453 & 0 \\ -2710 & 0 \\ -6000 & 0 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{C}^{\#} = (\underline{C}\ \underline{C}^{T})^{-1}\ \underline{C} = \left([\ 1\ 0\ ] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)^{-1} [\ 1\ 0\ ] = (1)\ [\ 1\ 0\ ] = [\ 1\ 0\ ]\ .$$

Next

$$\underline{\psi} = (\underline{\overline{T}}_{12} + \underline{\Sigma}\ \underline{C})(\underline{A}\ \underline{C}^{\#^T} - \underline{\dot{C}}^{\#^T}) - (\underline{\mathcal{D}} + \underline{\Sigma}\ \underline{\mathcal{H}})\ \underline{\Sigma} + \underline{\dot{\Sigma}}\ .$$

Now

$$\underline{\dot{C}}^{\#^T} = \underline{\dot{\Sigma}} = \underline{0}$$

thus

$$\underline{\psi} = \begin{bmatrix} -34 & 1 \\ -453 & 0 \\ -2710 & 0 \\ -6000 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) - \begin{bmatrix} -34 & 1 & 0 & 0 \\ -453 & 0 & 1 & 0 \\ -2710 & 0 & 0 & 1 \\ -6000 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -34 \\ -453 \\ -2710 \\ -6000 \end{bmatrix}$$

$$= \underline{0} - \begin{bmatrix} 703 \\ 12692 \\ 86140 \\ 204000 \end{bmatrix} \quad \Rightarrow \quad \underline{\psi} = \begin{bmatrix} -703 \\ -12692 \\ -86140 \\ -204000 \end{bmatrix} .$$

Having evaluated these integrals, $\dot{\underline{\S}}$ can now be found as

$$\dot{\underline{\S}} = (\underline{\mathcal{D}} + \underline{\Sigma}\ \underline{H})\ \underline{\S} + \underline{\psi}\ \underline{y} + \underline{\Omega}\ \underline{u}$$

$$= \begin{bmatrix} -34 & 1 & 0 & 0 \\ -453 & 0 & 1 & 0 \\ -2710 & 0 & 0 & 1 \\ -6000 & 0 & 0 & 0 \end{bmatrix} \underline{\S} + \begin{bmatrix} -703 \\ -12692 \\ -86140 \\ -204000 \end{bmatrix} \underline{y} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \underline{u} .$$

Given this, $\hat{\underline{x}}$ can be calculated from

$$\hat{\underline{x}} = [\ \underline{C}^T (\underline{C}\ \underline{C}^T)^{-1} - \underline{T}_{12}\ \underline{\Sigma}\ ]\ \underline{y} + \underline{T}_{12}\ \underline{\S}$$

$$= \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} [1] - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -34 \\ -453 \\ -2710 \\ -6000 \end{bmatrix} \right) \underline{y} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \underline{\S}$$

$$= \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -34 \end{bmatrix} \right) \underline{y} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \underline{\S} .$$

Finally,

$$\hat{\underline{x}} = \begin{bmatrix} 1 \\ 34 \end{bmatrix} \underline{y} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \underline{\S} \tag{44}$$

and

$$\underline{\hat{z}} = \underline{T}_{22} \, (\underline{\S} - \underline{\Sigma} \, \underline{y})$$

$$\underline{\hat{z}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left(\underline{\S} - \begin{bmatrix} -34 \\ -453 \\ -2710 \\ -6000 \end{bmatrix} \underline{y} \right) . \tag{45}$$

So for the reconstructor

$$\hat{x}_1 = y$$

$$\hat{x}_2 = 34y + \S_1$$

$$\hat{z}_1 = \S_2 + 453y$$

$$\hat{z}_2 = \S_3 + 2710y$$

$$\hat{z}_3 = \S_4 + 6000y \quad .$$

A diagram of this reduced-order observer is given in Figure 14.

A. DISCUSSION OF SIMULATION RESULTS

This case was simulated on a digital computer. The simulation program is shown in Appendix A along with a sample output. The results of a simulation are shown in Figure 15. For the results shown, the system output, y, was subjected to an initial condition of five units. The disturbance coefficients used are presented in Figure 16 and the total disturbance magnitude in Figure 17.

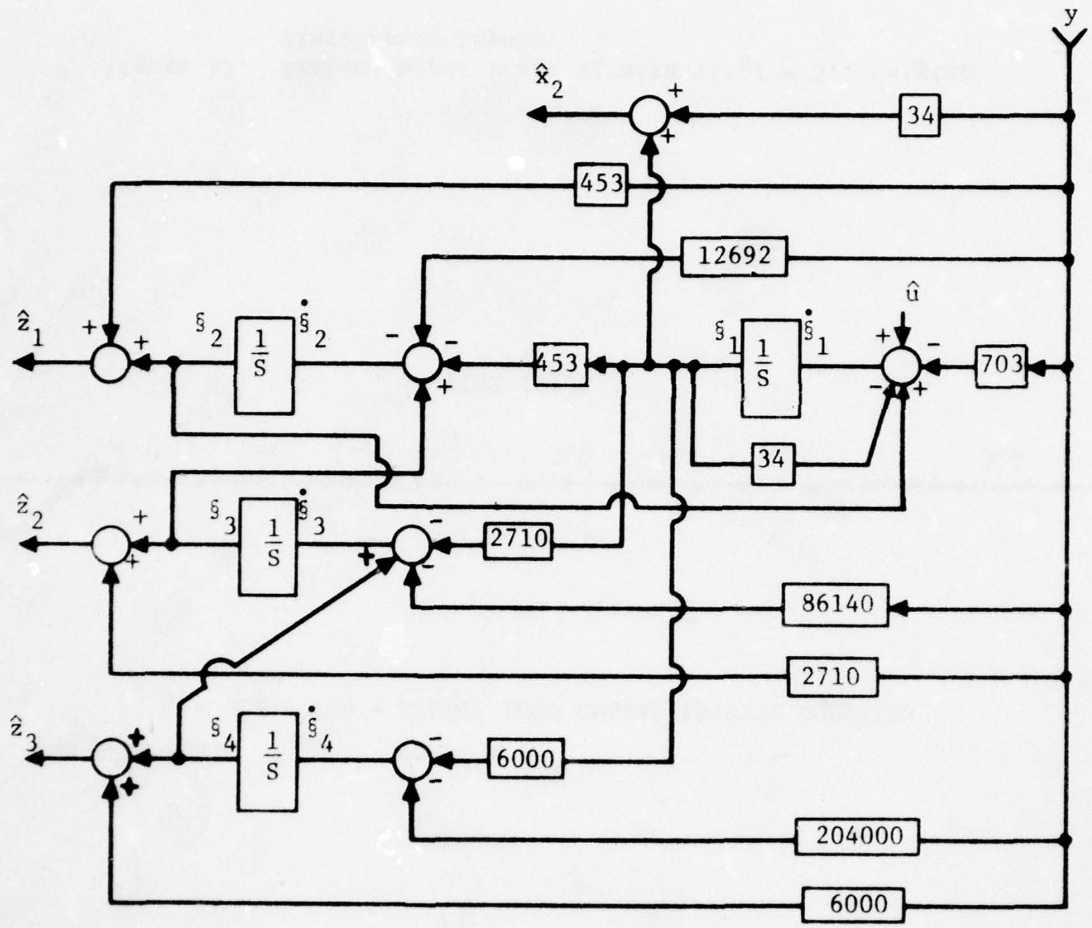


Figure 14. Reduced-order state reconstructor design, Case 2.

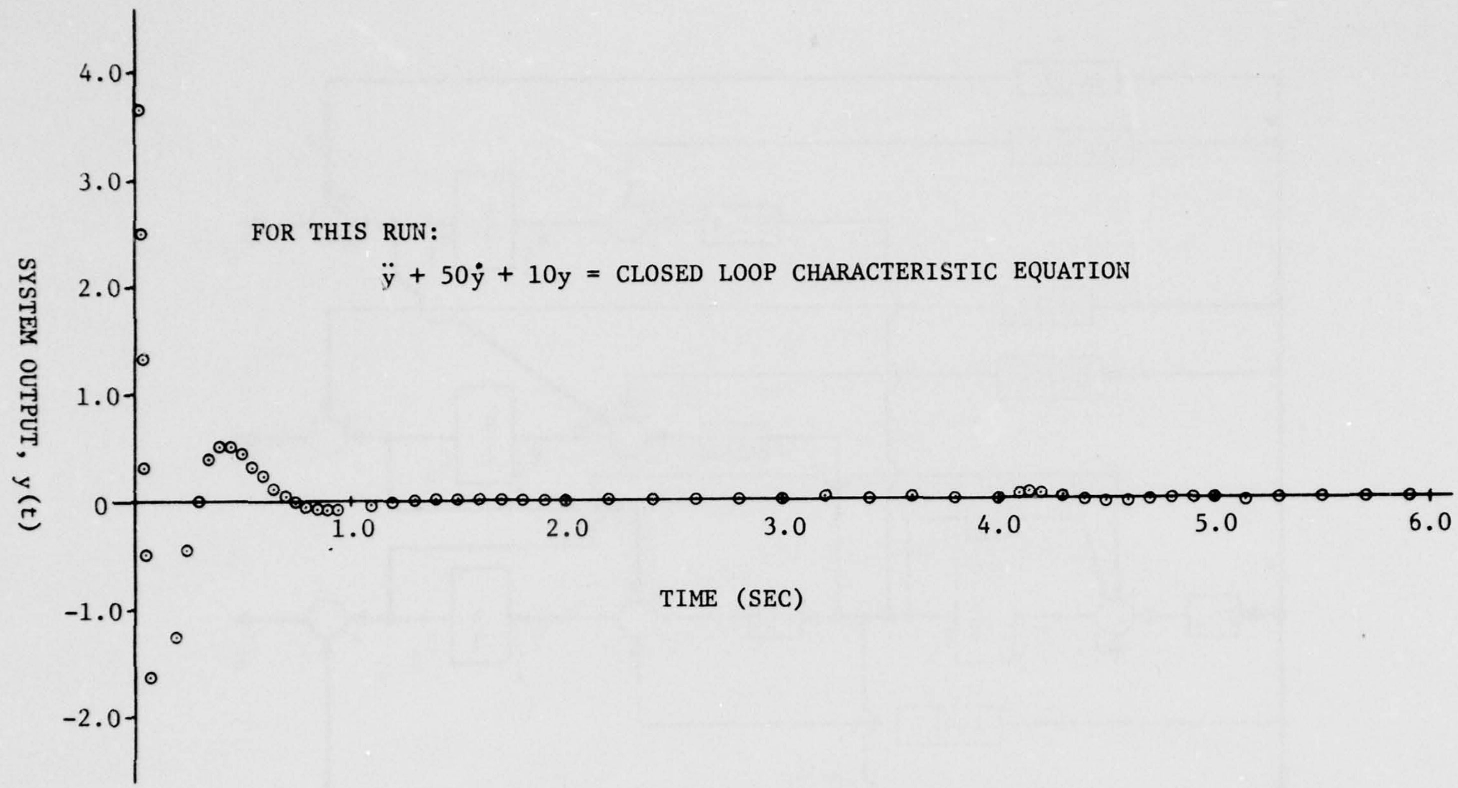


Figure 15. System output (Case 2) with $y(t_o) = 5.$, varying disturbance inputs.

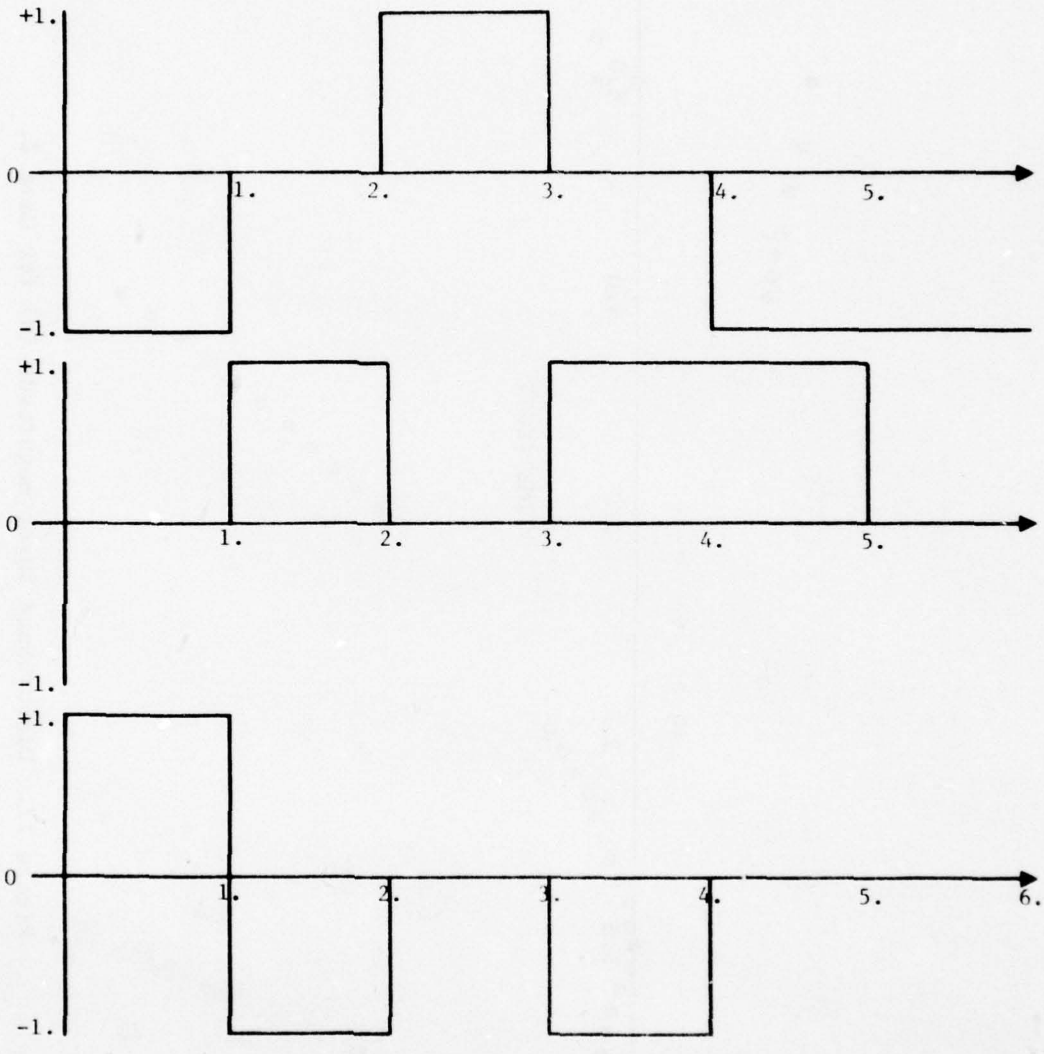


Figure 16. Disturbance input coefficients for Example No. 2.

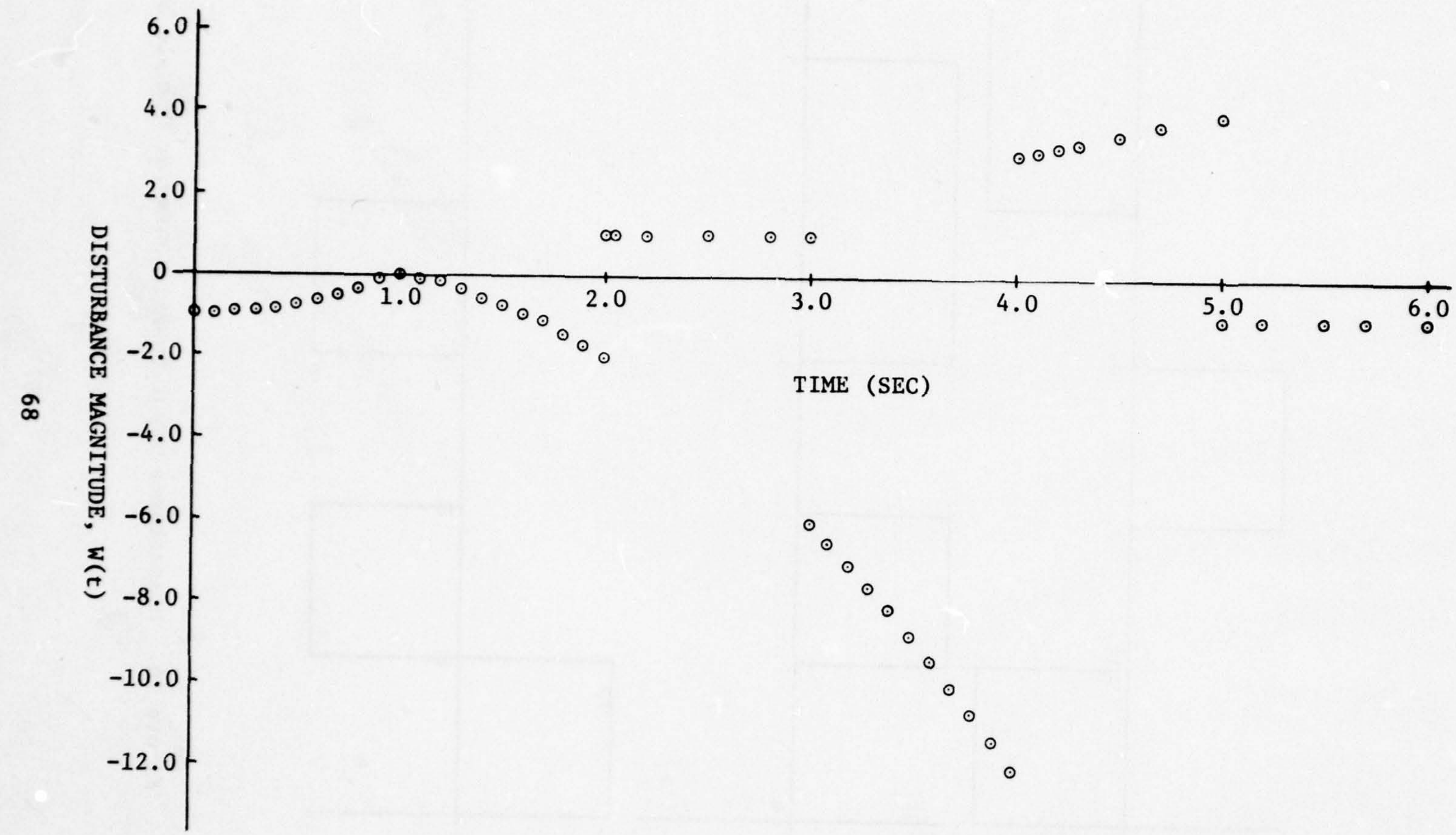
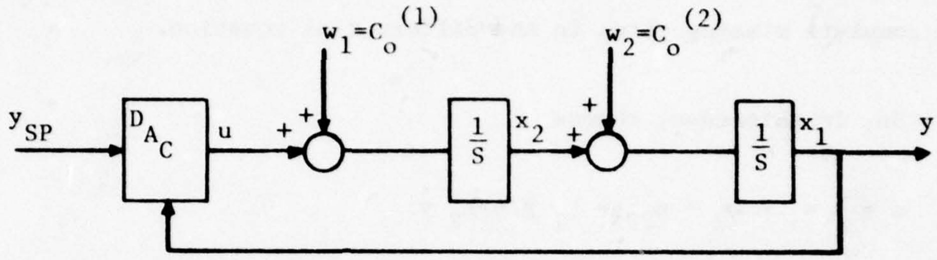


Figure 17. Disturbance input magnitude (W) for Case 2.

It is evident from Figure 15 that the disturbance input effects are being accommodated nicely.

5. EXAMPLE DESIGN PROBLEM NO. 3

The last example design problem is of the set-point regulator type for a second-order plant with vector disturbance as shown below.



The set-point input, y_{SP}, is a given constant and the disturbance input components, w_1 and w_2, are of the unknown piecewise constant type. The control design objective here is to make $y \to y_{SP}$ in the face of the given disturbance inputs.

To begin, the differential equation governing y(t) must be derived. From the given plant

$$y = x_1 \tag{46}$$

$$\left.\begin{aligned} \dot{x}_1 &= x_2 + w_2 \\ \dot{x}_2 &= u + w_1 \quad . \end{aligned}\right\} \tag{47}$$

Therefore

$$\dot{y} = \dot{x}_1 = x_2 + w_2 \tag{48}$$

$$\ddot{y} = \ddot{x}_1 = \dot{x}_2 + \dot{w}_2 = u + w_1 + \dot{w}_2 \quad . \tag{49}$$

Choose

$$u = f(w_1 , w_2 , \dot{w}_1 , \dot{w}_2 , y , \dot{y}) \tag{50}$$

such that y(t) consistently approaches y_{SP} for all possible disturbances, i.e., choose u to eliminate disturbance forms and derivatives and complete missing terms in the differential equation.

So, in this case, choose

$$u = - w_1 - \dot{w}_2 + u_{SP} - k_1 y - k_2 \dot{y} \tag{51}$$

thus

$$\ddot{y} = - w_1 - \dot{w}_2 + u_{SP} - k_1 y - k_2 \dot{y} + w_1 + \dot{w}_2$$

$$= - k_1 y - k_2 \dot{y} + u_{SP}$$

and

$$\ddot{y} + k_2 \dot{y} + k_1 y = u_{SP} \quad (k_1 > 0, k_2 > 0) \quad . \tag{52}$$

It is desired that u_{SP} be chosen such that

$$\lim_{t \to \infty} y(t) = y_{SP} \quad .$$

To find a suitable u_{SP}, proceed as follows: Let

$$\varepsilon_{SP} = \text{set-point error} = y_{SP} - y$$

then

$$\dot{\varepsilon}_{SP} = - \dot{y}$$

$$\ddot{\varepsilon}_{SP} = - \ddot{Y} = - u_{SP} + k_2 \dot{y} + k_1 y$$

$$= - u_{SP} + k_2 (- \dot{\varepsilon}_{SP}) + k_1 (y_{SP} - \varepsilon_{SP})$$

$$= - k_2 \dot{\varepsilon}_{SP} - k_1 \varepsilon_{SP} + k_1 y_{SP} - u_{SP} \quad . \tag{53}$$

It is desired that $\varepsilon_{SP}(t) \to 0$ quickly, so let

$$u_{SP} = k_1 y_{SP}$$

then

$$\ddot{\varepsilon}_{SP} + k_2 \dot{\varepsilon}_{SP} + k_1 \varepsilon_{SP} = 0 \quad . \tag{54}$$

Laplace transforming, we obtain

$$S^2 + k_2 S + k_1 = 0 \quad . \tag{55}$$

A second order system should factor as

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

thus giving

$$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0 \quad . \tag{56}$$

Comparing Equations (55) and (56), we see that

$$k_1 = \lambda_1\lambda_2$$

$$k_2 = -(\lambda_1 + \lambda_2) \quad .$$

Hence, choose λ_1 and λ_2 to be "large" negative numbers, say

$$\lambda_1 = -5$$

$$\lambda_2 = -10$$

thus

$$k_1 = 50$$

$$k_2 = 15 \quad .$$

This results in

$$u_{SP} = k_1 \, y_{SP} = 50 \, y_{SP}$$

and

$$u = - w_1 - \dot{w}_2 + u_{SP} - k_1 \, y - k_2 \, \dot{y}$$

$$= - w_1 - \dot{w}_2 + 50 \, y_{SP} - 50 \, y - 15 \, \dot{y}$$

$$= - w_1 + 50 \, y_{SP} - 50 \, x_1 - 15 \, x_2 \quad . \tag{57}$$

This u is the total value, i.e.,

$$u = u_R + u_c \quad . \tag{58}$$

To derive the full-dimensional state reconstructor, we start with the state equations for the plant and disturbances.

Plant

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \underline{u} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{59}$$

$$\underline{y} = (1 \;\; 0) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

thus

$$\underline{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \; ; \; \underline{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; ; \; \underline{F} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \; ; \; \underline{C} = [\, 1 \;\; 0 \,] \; .$$

Disturbances

$$w_1 = z_1 \qquad\qquad w_2 = z_2$$

$$\dot{z}_1 = \sigma_1 \qquad\qquad \dot{z}_2 = \sigma_2$$

$$\underline{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{60}$$

giving

$$\underline{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; ; \; \underline{D} \equiv \underline{M} \equiv \underline{L} \equiv \underline{0} \; .$$

Now, proceed as in Case 1 for the full-dimensional state reconstructor (Reference 1).

The observer has the form

$$\begin{pmatrix} \dot{\hat{\underline{x}}} \\ \dot{\hat{\underline{z}}} \end{pmatrix} = \left[\begin{array}{c|c} \underline{A} + \underline{F}\ \underline{L} + \underline{K}_1\ \underline{C} & \underline{F}\ \underline{H} \\ \hline \underline{M} + \underline{K}_2\ \underline{C} & \underline{D} \end{array}\right] \begin{pmatrix} \hat{\underline{x}} \\ \hat{\underline{z}} \end{pmatrix} - \begin{bmatrix} \underline{K}_1 \\ \underline{K}_2 \end{bmatrix} \underline{y} + \begin{bmatrix} \underline{B} \\ \underline{0} \end{bmatrix} \underline{u} \ .$$

For this example, this reduces to

$$\begin{pmatrix} \dot{\hat{\underline{x}}} \\ \dot{\hat{\underline{z}}} \end{pmatrix} = \left[\begin{array}{c|c} \underline{A} + \underline{K}_1\ \underline{C} & \underline{F}\ \underline{H} \\ \hline \underline{K}_2\ \underline{C} & \underline{0} \end{array}\right] \begin{pmatrix} \hat{\underline{x}} \\ \hat{\underline{z}} \end{pmatrix} - \begin{bmatrix} \underline{K}_1 \\ \underline{K}_2 \end{bmatrix} \underline{y} + \begin{bmatrix} \underline{B} \\ \underline{0} \end{bmatrix} \underline{u} \ .$$

For the error dynamics

$$\dot{\underline{\varepsilon}} = \left[\begin{array}{c|c} \underline{A} + \underline{K}_1\ \underline{C} & \underline{F}\ \underline{H} \\ \hline \underline{K}_2\ \underline{C} & \underline{0} \end{array}\right] (\underline{\varepsilon}) + \begin{pmatrix} \underline{0} \\ \underline{\sigma} \end{pmatrix}$$

with the given quantities

$$\underline{A} + \underline{K}_1\ \underline{C} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{pmatrix} k_{11} \\ k_{12} \end{pmatrix} (1\ \ 0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} k_{11} & 0 \\ k_{12} & 0 \end{bmatrix} = \begin{bmatrix} k_{11} & 1 \\ k_{12} & 0 \end{bmatrix}$$

$$\underline{F}\ \underline{H} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\underline{K}_2\ \underline{C} = \begin{pmatrix} k_{21} \\ k_{22} \end{pmatrix} (1\ \ 0) = \begin{bmatrix} k_{21} & 0 \\ k_{22} & 0 \end{bmatrix} \ .$$

Therefore

$$\dot{\underline{\varepsilon}} = \begin{bmatrix} k_{11} & 1 & 0 & 1 \\ k_{12} & 0 & 1 & 0 \\ k_{21} & 0 & 0 & 0 \\ k_{22} & 0 & 0 & 0 \end{bmatrix} (\underline{\varepsilon}) + \begin{pmatrix} \underline{0} \\ \underline{\sigma} \end{pmatrix} . \tag{61}$$

To determine $\underline{K}_1$, $\underline{K}_2$, let

$$\underline{\tilde{A}} = \begin{bmatrix} k_{11} & 1 & 0 & 1 \\ k_{12} & 0 & 1 & 0 \\ k_{21} & 0 & 0 & 0 \\ k_{22} & 0 & 0 & 0 \end{bmatrix} = \text{characteristic matrix of the } \dot{\varepsilon} \text{ dynamics.}$$

It is desired that $\varepsilon(t) \to 0$ rapidly; therefore, the roots of the characteristic polynomial should be "large" negative numbers. To find the eigenvalues of $\underline{\tilde{A}}$, we proceed as follows:

$$|\underline{\tilde{A}} - \lambda \underline{I}| = \begin{vmatrix} (k_{11} - \lambda) & 1 & 0 & 1 \\ k_{12} & -\lambda & 1 & 0 \\ k_{21} & 0 & -\lambda & 0 \\ k_{22} & 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$(k_{11} - \lambda)(-\lambda^3) - 1\,(\lambda^2 k_{12} + k_{21}\,\lambda) - 1\,(\lambda^2 k_{22}) = 0$$

$$\lambda^4 - k_{11}\,\lambda^3 - k_{12}\,\lambda^2 - k_{22}\,\lambda^2 - k_{21}\,\lambda = 0$$

$$\lambda^3 - k_{11}\lambda^2 - (k_{12} + k_{22})\lambda - k_{21} = 0 \quad . \tag{62}$$

Let the roots be λ_1, λ_2, λ_3. Then

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0$$

$$(\lambda - \lambda_1)[\lambda^2 - (\lambda_2 + \lambda_3)\lambda + \lambda_2\lambda_3] = 0$$

so

$$\lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda - \lambda_1\lambda_2\lambda_3 = 0 \quad . \tag{63}$$

Comparing Equations (62, 63), we see that

$$-k_{11} = -(\lambda_1 + \lambda_2 + \lambda_3) \Rightarrow k_{11} = \lambda_1 + \lambda_2 + \lambda_3$$

$$-k_{21} = -\lambda_1\lambda_2\lambda_3 \Rightarrow k_{21} = \lambda_1\lambda_2\lambda_3$$

$$-k_{12} - k_{22} = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \Rightarrow k_{12} + k_{22} = -(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) \quad .$$

Let

$$\lambda_1 = -5$$
$$\lambda_2 = -8 + j4$$

$$\lambda_3 = -8 - j4 \quad .$$

Then

$$k_{11} = -5 - 8 + j4 - 8 - j4 = -21$$

$$k_{21} = -5\,[80] = -400$$

$$k_{12} + k_{22} = -\,[40 - 5j4 + 40 + 5j4 + 80] = -160$$

$$k_{12} = -160 - k_{22} \quad .$$

Let

$$k_{22} = -80, \text{ then } k_{12} = -80 \quad .$$

Thus

$$\underline{K}_1 = \begin{pmatrix} -21 \\ -80 \end{pmatrix} \qquad \underline{K}_2 = \begin{pmatrix} -400 \\ -80 \end{pmatrix} \quad .$$

Substituting back into the expression for the reconstructor

$$\begin{pmatrix} \dot{\hat{\underline{x}}} \\ \dot{\hat{\underline{z}}} \end{pmatrix} = \begin{bmatrix} -21 & 1 & 0 & 1 \\ -80 & 0 & 1 & 0 \\ -400 & 0 & 0 & 0 \\ -80 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{z}_1 \\ \hat{z}_2 \end{pmatrix} - \begin{bmatrix} -21 \\ -80 \\ -400 \\ -80 \end{bmatrix} \underline{y} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u \qquad (64)$$

and

$$\dot{\hat{x}}_1 = -21\,\hat{x}_1 + \hat{x}_2 + \hat{z}_2 + 21\,y$$

$$\dot{\hat{x}}_2 = - 80\ \hat{x}_1 + \hat{z}_1 + 80\ y + \hat{u}$$

$$\dot{\hat{z}}_1 = - 400\ \hat{x}_1 + 400\ y$$

$$\dot{\hat{z}}_2 = - 80\ \hat{x}_1 + 80\ y \quad .$$

Thus

$$u = 50\ y_{SP} - 50\ \hat{x}_1 - 15\ \hat{x}_2 - 15\ \hat{z}_2 - \hat{z}_1 \tag{65}$$

is the sought control.

A block diagram of the system with the state reconstructor is shown in Figure 18. This problem was programmed on an EAI-221 analog computer for checkout of the results. Table 1 lists the programmed variables, the maximum values expected and the scaled value for each. Appendix B shows the scaling process for programming the analog and Figure 19 gives the patching diagram for the problem.

A. DISCUSSION OF ANALOG SIMULATION RESULTS

Figure 20 shows the plant response to individual inputs y_{SP}, w_1, and w_2. Response is good to changes in the setpoint and to changes in w_2. Response to w_1 is more sluggish. Figure 21 further emphasizes the sluggish response to w_1 but indicates that the disturbance is nulled out when given time between impulses in w_1.

Figure 22 has output for simultaneous applications of w_1 and w_2 with an off-zero setpoint. Note that when time between impulses in the disturbances is allowed, the disturbance effect is nulled out.

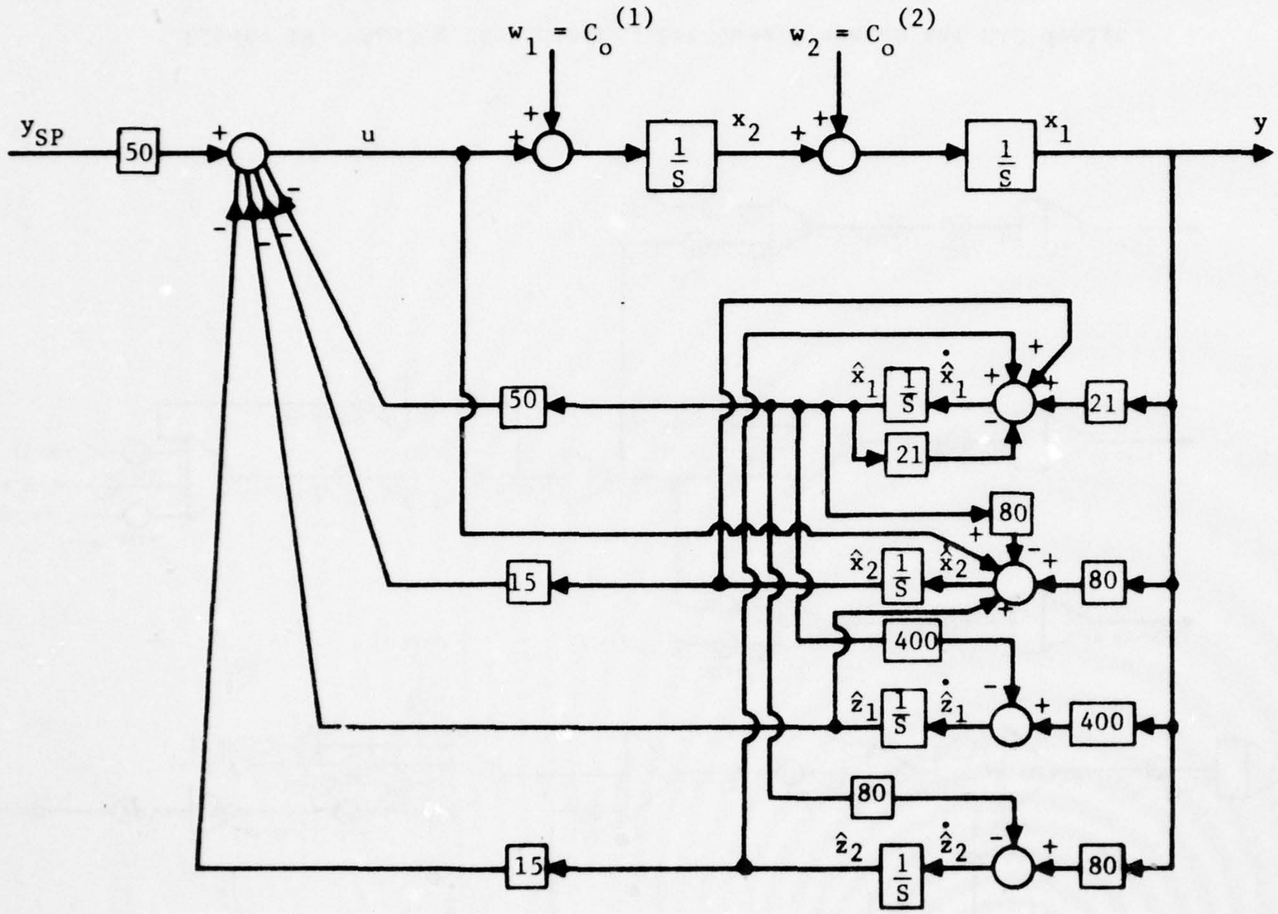


Figure 18. System with DAC and full-dimensional state reconstructor, Case 3.

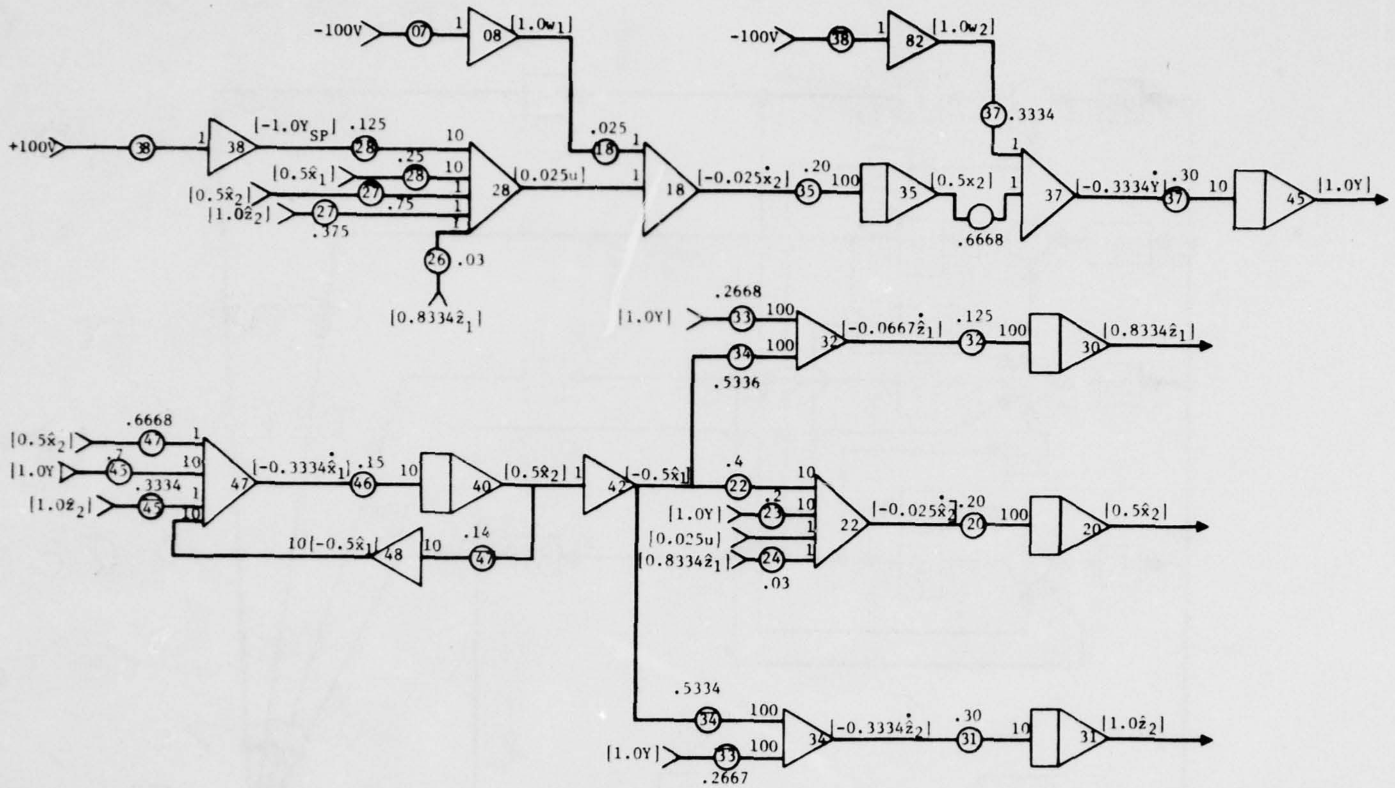


Figure 19. Analog flow diagram for Case 3 system and DAC design.

TABLE 1. SCALED VARIABLES FOR ANALOG SIMULATION, PROBLEM NO. 3		
VARIABLE	MAXIMUM VALUE	SCALED VARIABLE
w_1	100	$[\ 1.0\ w_1\ ]$
w_2	100	$[\ 1.0\ w_2\ ]$
y_{SP}	100	$[\ 1.0\ y_{SP}\ ]$
$\dot{\hat{x}}_1$	300	$[\ 0.3334\ \dot{\hat{x}}_1\ ]$
$\hat{x}_1$	200	$[\ 0.500\ \hat{x}_1\ ]$
$\dot{\hat{x}}_2$	4000	$[\ 0.025\ \dot{\hat{x}}_2\ ]$
$\hat{x}_2$	200	$[\ 0.500\ \hat{x}_2\ ]$
$\dot{\hat{z}}_1$	1500	$[\ 0.06667\ \dot{\hat{z}}_1\ ]$
$\hat{z}_1$	120	$[\ 0.8334\ \hat{z}_1\ ]$
$\dot{\hat{z}}_2$	300	$[\ 0.334\ \dot{\hat{z}}_2\ ]$
$\hat{z}_2$	100	$[\ 1.0\ \hat{z}_2\ ]$
$y(x_2)$	100	$[\ 1.0\ y\ ]$
u	4000	$[\ 0.025\ u\ ]$
x_1	200	$[\ 0.500\ x_1\ ]$
$\dot{x}_2$	4000	$[\ 0.025\ \dot{x}_1\ ]$
$\dot{y}(\dot{x}_1)$	300	$[\ 0.3334\ \dot{y}\ ]$

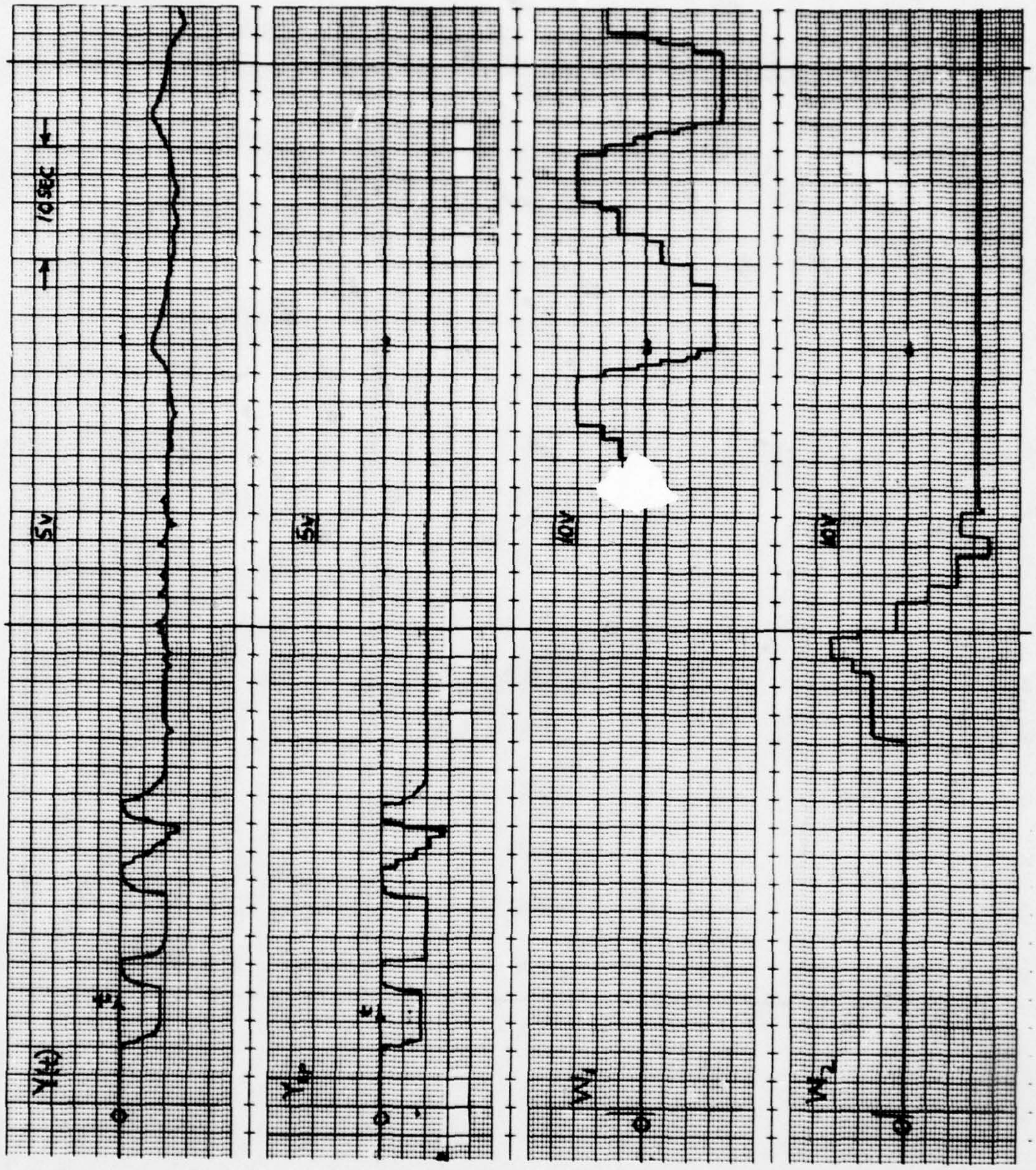


Figure 20. Output response to changes in y_{SP} and to individual w_1 and w_2 inputs.

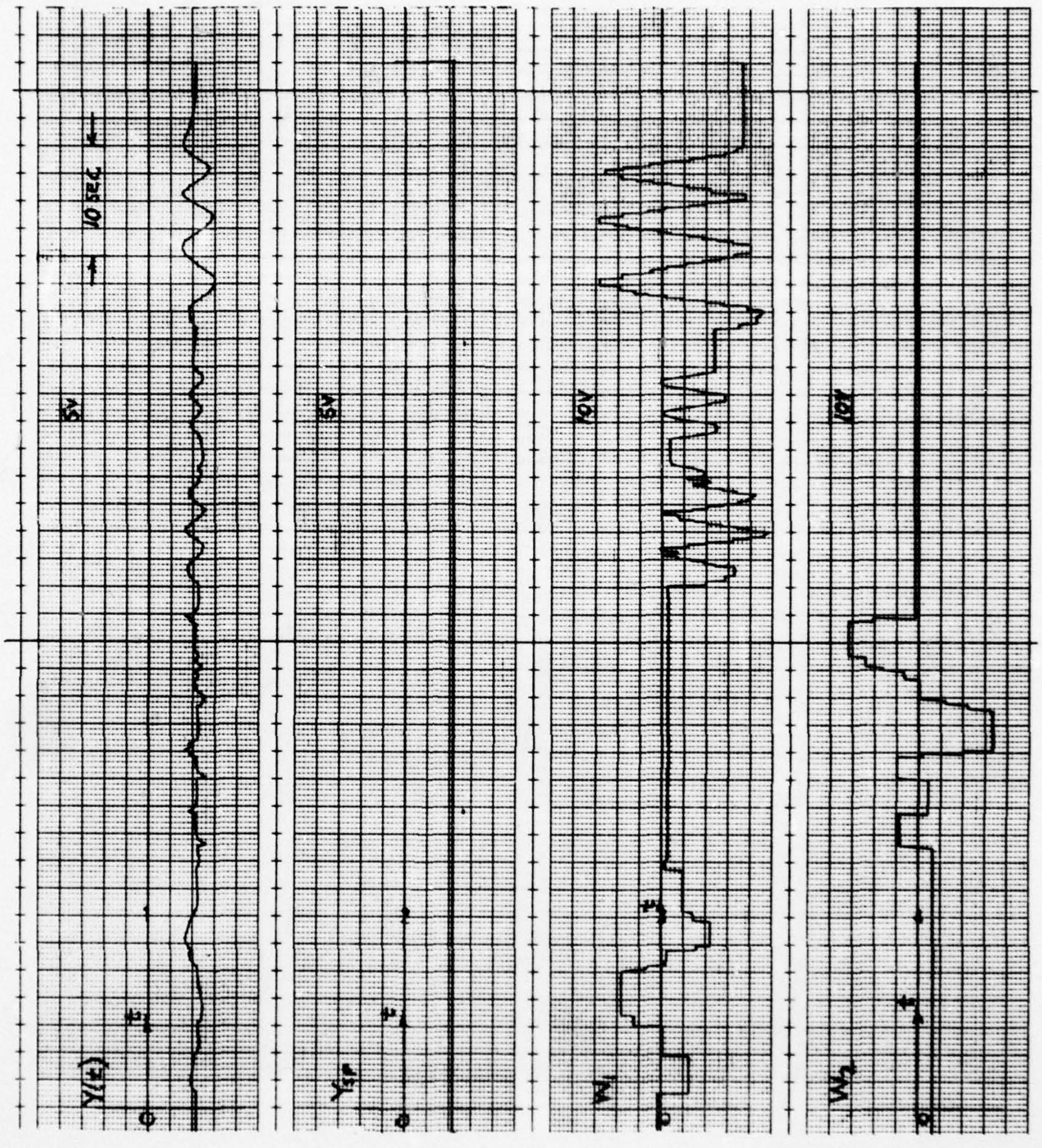


Figure 21. Output response to individual w_1 and w_2 inputs with given y_{SP}.

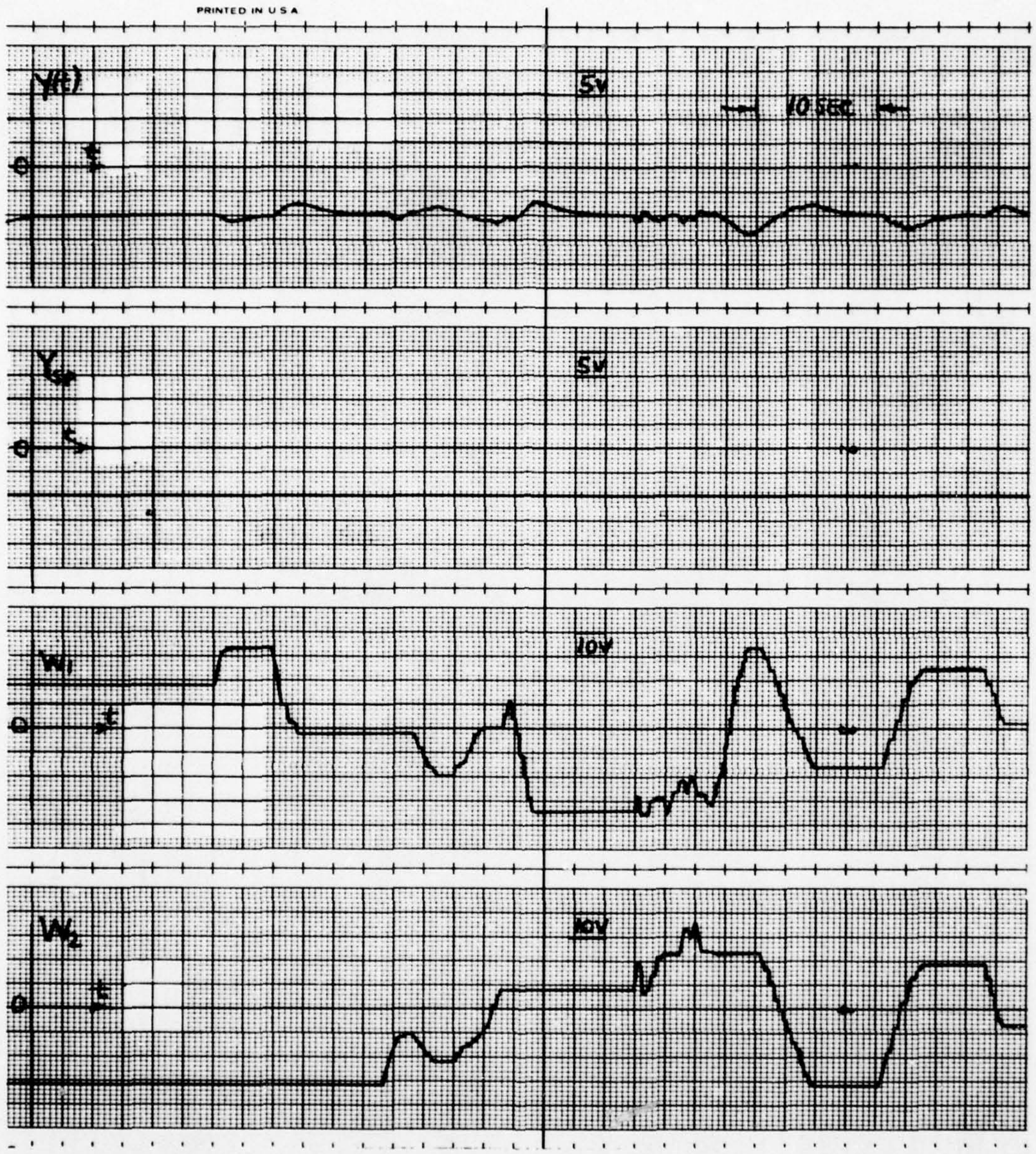


Figure 22. Output response to simultaneous applications of w_1 and w_2.

6. CONCLUSIONS

The DACs arrived at for each example (using the methods developed in the references) performed, in each case, those functions which they were designed to perform. The effects of unmeasurable disturbance inputs were canceled out by the portion of the control which was designed specifically to handle a given waveform mode disturbance. When the impulse train was not of too high a frequency, the errors engendered by the disturbances were settled out very well.

APPENDIX A

APPENDIX A

This appendix contains a listing of the computer simulation of Example No. 2 and a sample output. In this program, the following symbols apply:

X1 = output of first integrator after u + w

X2 = Y = system output

X3 = $\S_1$

X4 = $\S_2$

X5 = $\S_3$

X6 = $\S_4$

Z1 = $\hat{Z}_1$

```
PROGRAM MAIN   74/74   OPT=1     FTN4.2+74355   04/21/78   10.59.20.

                                                                PAGE 1

          PROGRAM MAIN(INPUT, OUTPUT, TAPE5=INPUT, TAPE6=OUTPUT)
C         INTEGRALS FROM RUNGK
          COMMON X1, X2, X3, X4, X5, X6
C         DERIVATIVES TO RUNGK
          COMMON XD1, XD2, XD3, XD4, XD5, XD6
          COMMON KUTTA,DT,NX
          J=1
          ALF1=1Ø.
          ALF2=5Ø.
          NX=6
          DT=.Ø1
          XD1=XD2=XD3=XD4=XD5=XD6=Ø.
          X1=X3=X4=X5=X6=Ø.
          X2=5.
          W=U=UC=UR=Ø.
          WRITE (6,2ØØØ)
     2ØØØ FORMAT (2X,*TIME*,12X,*Y*,15X,*W*,15X,*U*,/)
     1Ø1Ø CONTINUE
          IF(TIME.GE.1.) GO TO 1Ø
          CØ=-1.
          C1=Ø.
          C2=1.
          GO TO 5Ø
       1Ø IF(TIME.GE.2.) GO TO 11
          CØ=Ø.
          C1=1.
          C2=-1.
          GO TO 5Ø
       11 IF(TIME.GE.3.) GO TO 12
          CØ=1.
          C1=Ø.
          C2=Ø.
          GO TO 5Ø
       12 IF(TIME.GE.4.) GO TO 13
          CØ=Ø.
          C1=1.
          C2=-1.
          GO TO 5Ø
       13 IF(TIME.GE.5.) GO TO 14
          CØ=-1.
          C1=1.
          C2=Ø.
          GO TO 5Ø
       14 CØ=1.
          C1=Ø.
          C2=Ø.
```

```
   50 CONTINUE
      IF(TIME.GE.6.) GO TO 1000
      DO 100 KUTTA=1,4
      W=C0+C1*TIME+C2*TIME*TIME
      U=UC+UR
      XD1=W+U
      XD2=X1
      XD3=-703.*X2-34.*X3+U+X4
      XD4=-453.*X3-12692.*X2+X5
      XD5=-86140.*X2-2710.*X3+X4
      XD6=-6000.*X3-204000.*X2
      Z1=X4+453.*X2
      UC=-Z1
      UR=-ALF1*X2-ALF2*(X3+34.*X2)
      Y=X2
      GO TO (30,60,30,40),KUTTA
   30 TIME=TIME+.5*DT
   40 CONTINUE
   60 CALL RUNGK
  100 CONTINUE
      WRITE (6,2001)TIME,Y,W,U
 2001 FORMAT (2X,4(G12.6,3X)
      GO TO 1010
 1000 CONTINUE
      END
```

```
      SUBROUTINE RUNGK
      COMMON X,DX,KUTTA,DT,NX
      DIMENSION X(6),DX(6),XA(6),DXA(6)
      GO TO (10,30,50,70),KUTTA
   10 DO 20 I=1,NX
      XA(I)=X(I)
      DXA(I)=DT*DX(I)
   20 X(I)=X(I)+.5*DXA(I)
      RETURN
   30 TDT=2.*DT
      HDT=.5*DT
      DO 40 I=1,NX
      DXA(I)=DXA(I)+TDT*DX(I)
   40 X(I)=XA(I)+HDT*DX(I)
      RETURN
   50 DO 60 I=1,NX
      VDT=DT*DX(I)
      DXA(I)=DXA(I)+2.*VDT
   60 X(I)=XA(I)+VDT
      RETURN
   70 DO 80 I=1,NX
   80 X(I)=XA(I)+(DXA(I)+DT*DX(I))/6.
      RETURN
      END
```

```
 $$$$$$$$  $$$$$$$     $$     $$$$$$   $     $$  $$$$$$$  $      $
    $$     $$    $$   $$$    $$    $$  $$    $$  $$    $$  $$    $$
    $$     $$    $$    $$    $$        $$$   $$        $$  $$$  $$$
    $$     $$$$$$$     $$    $$        $$$$  $$    $$$$$   $$$$$$$$
    $$     $$    $$    $$    $$  $$$$  $$ $$ $$   $$       $$ $$ $$
    $$     $$    $$    $$    $$    $$  $$  $$$$  $$        $$    $$
    $$     $$    $$    $$    $$    $$  $$   $$$  $$        $$    $$
$$$$$$$$$  $$$$$$$  $$$$$$$$  $$$$$$   $$    $$  $$$$$$$$  $$    $$
```

TIME	Y	W	U
.500000E-01	.326144	-.997500	1736.07
.100000E+00	-2.20212	-.990000	1064.53
.150000	-2.09702	-.977500	278.066
.200000	-1.24473	-.960000	-31.3571
.250000	-.442264	-.937500	-109.358
.300000	.987918E-01	-.910000	-101.923
.350000	.388633	-.877500	-72.7892
.400000	.496384	-.840000	-44.9627
.450000	.490198	-.797500	-24.2326
.500000	.421491	-.750000	-10.3340
.550000	.324981	-.697500	-1.55719
.600000	.222706	-.640000	3.69061
.650000	.127913	-.577500	6.55314
.700000	.478995E-01	-.510000	7.79105
.750000	-.140812E-01	-.437500	7.91664
.800000	-.575379E-01	-.360000	7.29715
.850000	-.838194E-01	-.277500	6.21361
.900000	-.954126E-01	-.190000	4.88815
.950000	-.953897E-01	-.975000E-01	3.49451
1.00000	-.869770E-01	-.312639E-12	2.16203
1.05000	-.732859E-01	-.525000E-01	1.02897
1.10000	-.571629E-01	-.110000	.235651
1.15000	-.407288E-01	-.172500	-.292992
1.20000	-.254524E-01	-.240000	-.610963
1.25000	-.122873E-01	-.312500	-.750882
1.30000	-.175652E-02	-.390000	-.742044
1.35000	.597311E-02	-.472500	-.615799
1.40000	.110129E-01	-.560000	-.404780
1.45000	.136710E-01	-.652500	-.140298
1.50000	.143752E-01	-.750000	.150097
1.55000	.136042E-01	-.852500	.444192
1.60000	.118334E-01	-.960000	.725774
1.65000	.949435E-02	-1.07250	.984509
1.70000	.694935E-02	-1.19000	1.21529
1.75000	.447902E-02	-1.31250	1.41729
1.80000	.228023E-02	-1.44000	1.59283
1.85000	.471931E-03	-1.57250	1.74638
1.90000	-.893733E-03	-1.71000	1.88351

1.95000	-.181761E-02	-1.85250	2.01015
2.00000	-.233887E-02	-2.00000	2.13192
2.05000	-.415149E-03	1.00000	.231780
2.10000	.435080E-02	1.00000	-1.88795
2.15000	.718942E-02	1.00000	-2.16349
2.20000	.725833E-02	1.00000	-1.77639
2.25000	.543347E-02	1.00000	-1.32481
2.30000	.280351E-02	1.00000	-.996430
2.35000	.174617E-03	1.00000	-.810407
2.40000	-.199187E-02	1.00000	-.733965
2.45000	-.350440E-02	1.00000	-.728246
2.50000	-.434698E-02	1.00000	-.762274
2.55000	-.460272E-02	1.00000	-.814672
2.60000	-.440078E-02	1.00000	-.871784
2.65000	-.388231E-02	1.00000	-.925403
2.70000	-.317993E-02	1.00000	-.971005
2.75000	-.240644E-02	1.00000	-1.00654
2.80000	-.164973E-02	1.00000	-1.03162
2.85000	-.971774E-03	1.00000	-1.04694
2.90000	-.410388E-03	1.00000	-1.05389
2.95000	.173459E-04	1.00000	-1.05418
3.00000	.310808E-03	1.00000	-1.04964
3.05000	-.439831E-02	-6.25250	3.64249
3.10000	-.150811E-01	-6.51000	8.76494
3.15000	-.207983E-01	-6.77250	9.57097
3.20000	-.198250E-01	-7.04000	8.85373
3.25000	-.144049E-01	-7.31250	8.01744
3.30000	-.721390E-02	-7.59000	7.49761
3.35000	-.209722E-03	-7.87250	7.33350
3.40000	.550040E-02	-8.16000	7.44314
3.45000	.947118E-02	-8.45250	7.73122
3.50000	.116878E-01	-8.75000	8.12178
3.55000	.123785E-01	-9.05250	8.56214
3.60000	.118836E-01	-9.36000	9.01839
3.65000	.105715E-01	-9.67250	9.47000
3.70000	.878648E-02	-9.99000	9.90557
3.75000	.682119E-02	-10.3125	10.3199
3.80000	.490318E-02	-10.6400	10.7121
3.85000	.319274E-02	-10.9725	11.0840
3.90000	.178744E-02	-11.3100	11.4391
3.95000	.731072E-03	-11.6525	11.7820
4.00000	.247644E-04	-12.0000	12.1174
4.05000	.100493E-01	3.05000	2.45679
4.10000	.327259E-01	3.10000	-7.99785
4.15000	.446498E-01	3.15000	-9.16160
4.20000	.422447E-01	3.20000	-7.06363
4.25000	.304107E-01	3.25000	-4.72115
4.30000	.149235E-01	3.30000	-3.06286
4.35000	-.210559E-04	3.35000	-2.16605

4.40000	-.120745E-01	3.40000	-1.84939
4.45000	-.203173E-01	3.45000	-.190485
4.50000	-.247537E-01	3.50000	-2.16745
4.55000	-.259043E-01	3.55000	-2.52328
4.60000	-.245235E-01	3.60000	-2.89938
4.65000	-.214174E-01	3.65000	-3.25192
4.70000	-.173343E-01	3.70000	-3.55690
4.75000	-.129050E-01	3.75000	-3.80376
4.80000	-.861629E-02	3.80000	-3.99104
4.85000	-.480742E-02	3.85000	-4.12323
4.90000	-.168117E-02	3.90000	-4.20847
4.95000	.675875E-03	3.95000	-4.25674
5.00000	.226859E-02	4.00000	-4.27845
5.05000	-.274709E-03	-1.00000	-.980014
5.10000	-.740840E-02	-1.00000	2.51706
5.15000	-.112498E-01	-1.00000	2.89988
5.20000	-.105737E-01	-1.00000	2.20107
5.25000	-.696969E-02	-1.00000	1.42842
5.30000	-.230132E-02	-1.00000	.887047
5.35000	.210199E-02	-1.00000	.597906
5.40000	.552264E-02	-1.00000	.497356
5.45000	.770752E-02	-1.00000	.514587
5.50000	.869596E-02	-1.00000	.594484
5.55000	.868376E-02	-1.00000	.699791
5.60000	.793059E-02	-1.00000	.807349
5.65000	.670221E-02	-1.00000	.903846
5.70000	.523724E-02	-1.00000	.982534
5.75000	.373028E-02	-1.00000	1.04099
5.80000	.232582E-02	-1.00000	1.07958

AD-A060 628 ARMY MISSILE RESEARCH AND DEVELOPMENT COMMAND REDSTO--ETC F/G 9/3
INVESTIGATION OF DISTURBANCE ACCOMMODATING CONTROLLER DESIGN.(U)
JUL 78 W L MCCOWAN
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APPENDIX B

APPENDIX B

In order to program Example No. 3 on the analog without having to resort to very high gains on the amplifiers, a magnitude scaling process was used. Equations (47), (64) and (65) were programmed thusly with maximum expected values being normalized to 100 volts. The resulting expressions give the pot settings and amplifier gains for patching each of the equations on the analog.

$$\dot{\hat{x}}_1 = - 21\ \hat{x}_1 + \hat{x}_2 + \hat{z}_2 + 21\ y$$

$$[\ 0.3334\ \dot{\hat{x}}_1\ ] = \frac{(0.3334)(-\ 21)}{(0.5)}\ [(0.5)\ \hat{x}_1\ ] + \frac{(0.3334)}{0.5}\ [0.5\ \hat{x}_2\ ]$$

$$+ \frac{(0.3334)}{1.0}\ [1.0\ \hat{z}_2\ ] + \frac{(0.3334)(21)}{1}\ [1.0\ y]$$

$$[\ 0.3334\ \dot{\hat{x}}_1\ ] = - (.14)\ 100\ [\ 0.5\ \hat{x}_1\ ] + (.6668)\ [0.5\ \hat{x}_2\ ]$$

$$+ (0.3334)\ [1.0\ \hat{z}_2\ ] + (.7)\ 10\ [\ 1.0\ y]$$

$$\dot{\hat{x}}_2 = - 80\ \hat{x}_1 + \hat{z}_1 + 80\ y + u$$

$$[\ 0.025\ \dot{\hat{x}}_2\ ] = \frac{(0.025)(-\ 80)}{0.5}\ [\ 0.5\ \hat{x}_1\ ] + \frac{(0.025)}{0.8334}\ [\ 0.8334\ \hat{z}_1\ ]$$

$$+ \frac{(0.025)(80)}{1.0}\ [1.0\ y\ ] + \frac{(0.025)}{0.025}\ [\ 0.025\ u]$$

$$[\,0.025\ \dot{\hat{x}}_2\,] = -\,(.4)\ 10\,[\,0.5\ \hat{x}_1\,] + (0.3)\,[\,0.8334\ \hat{z}_1\,]$$

$$+\ (.2)\ 10\ [1.0\ y] + [0.025\ u]$$

$$\dot{\hat{z}}_1 = -\ 400\ \hat{x}_1 + 400\ y$$

$$[\,0.0667\ \dot{\hat{z}}_1\,] = \frac{(0.0667)(-\ 400)}{0.5}\,[0.5\ \hat{x}_1\,] + \frac{(0.0667)(400)}{1.0}\,[1.0\ y\,]$$

$$[\,0.0667\ \dot{\hat{z}}_1\,] = -\,(.5336)\ 100\,[\,0.5\ \hat{x}_1\,] + (.2668)\ 100\,[\,1.0\ y\,]$$

$$\dot{\hat{z}}_2 = -\ 80\ \hat{x}_1 + 80\ y$$

$$[\,0.3334\ \dot{\hat{z}}_2\,] = \frac{(0.3334)(-\ 80)}{0.5}\,[0.5\ \hat{x}_1\,] + \frac{(0.3334)(80)}{1.0}\,[\,1.0\ y\,]$$

$$[\,0.3334\ \dot{\hat{z}}_2\,] = -\,(.5334)\ 100\,[\,0.5\ \hat{x}_1\,] + (.2667)\ 100\,[\,1.0\ y\,]$$

$$u = 50\ y_{SP} - 50\ \hat{x}_1 - 15\ \hat{x}_2 - 15\ \hat{z}_2 - \hat{z}_1$$

$$[\,0.025\ u\,] = \frac{(0.025)(50)}{1.0}\,[\,1.0\ y_{SP}\,] - \frac{(0.025)(50)}{0.5}\,[0.5\ \hat{x}_1\,]$$

$$-\ \frac{(0.025)(15)}{0.5}\,[\,0.5\ \hat{x}_2\,] - \frac{(0.025)(15)}{1.0}\,[\,1.0\ \hat{z}_2\,]$$

$$- \frac{(0.025)}{0.8334} [\, 0.8334 \; \hat{z}_1 \,]$$

$$[\, 0.025 \; u \,] \quad = (.125) \; 10 \, [\, 1.0 \; y_{SP} \,] - (.25) \; 10 \, [\, 0.5 \; \hat{x}_1 \,]$$

$$- (.75) \, [\, 0.5 \; \hat{x}_2 \,] - (.375) \, [\, 1.0 \; \hat{z}_2 \,] - (0.3) \, [\, 0.8334 \; \hat{z}_1 \,]$$

$$\dot{x}_2 \quad = u + w_1$$

$$[\, 0.025 \; \dot{x}_2 \,] \quad = \frac{(0.025)}{0.025} [\, 0.025 \; u \,] + \frac{(0.025)}{1.0} [\, 1.0 \; w_1 \,]$$

$$[\, 0.025 \; \dot{x}_2 \,] \quad = [\, 0.025 \; u \,] + (.025) \, [\, 1.0 \; w_1 \,]$$

$$\dot{y} = x_2 + w_2$$

$$[\, 0.3334 \; \dot{y} \,] \quad = \frac{(0.3334)}{0.5} [\, 0.5 \; x_2 \,] + \frac{(0.3334)}{1.0} [\, 1.0 \; w_2 \,]$$

$$[\, 0.3334 \; \dot{y} \,] \quad = (.6668) \, [\, 0.5 \; x_2 \,] + (0.3334) \, [\, 1.0 \; w_2 \,]$$

REFERENCES

1. Johnson, C. D., "Accommodation of Disturbances in Optimal Control Problems," Int. J. Control, 1972, Volume 15, No. 2, 209-231.

2. Johnson, C. D., "On Observers for Systems with Unknown and Inaccessible Inputs," Int. J. Control, 1975, Volume 21, No. 5, 825-831.

3. Johnson, C. D., "Accommodation of External Disturbances in Linear Regulator and Servomechanism Problems," IEEE Trans. On Auto. Control, Volume AC-16, No. 6, December 1971.

4. Johnson, C. D., "Algebraic Solution of the Servomechanism Problem with External Disturbances," Trans. of ASME, J. Dyn. Sys., Meas. and Cont., March 1974, 25-35.

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